Representation theory & the Hubbard model

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Outline

1. The Hubbard model
2. Representation theory of the symmetric group S_n
3. Representation theory of the special unitary group $SU(n)$
4. Ordering of energy levels

The Hubbard model

- General Hamiltonian

$$H = \underbrace{-\sum_{i,j,\sigma} t_{ij} \left(c^{\dagger}_{i\sigma} c_{j\sigma} + \text{h.c.} \right)}_{\text{hopping term}} + \underbrace{\sum_{i} U_i (n_i - \mu)^2}_{\text{on-site interaction}}$$

$$i, j \in \Lambda, \qquad \sigma \in \{1, \ldots, N\}$$

- Simplifying assumptions

$$t_{ij} = t \text{ for nearest neighbours, } U_i = U \text{ for all sites}$$

Strong coupling limit

- Strong on-site interaction, weak (relative to on-site) hopping term:

$$\frac{U}{t} \gg 1$$

- Second order perturbation theory

$$H_{\text{eff}} = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \quad J = \frac{2t^2}{U}$$

(generalized) Heisenberg antiferromagnet, $\mathbf{S}_i$ is "spin" operator at site i: usual spin operator for $N = 2$, generalized spin operator for $N > 2$

Representation theory of the symmetric group S_n

- S_n = group of permutations of $\{1, \ldots, n\}$
- notation for $\sigma \in S_n$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

or in cycle notation

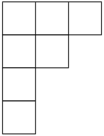
$$\begin{pmatrix} 12345 \\ 53124 \end{pmatrix} \leftrightarrow (15423), \quad \begin{pmatrix} 12345 \\ 31254 \end{pmatrix} \leftrightarrow (132)(45)$$

- Conjugacy class of $a \in S_n$

$$Cl(a) = \{b \in S_n : \exists\, g \in S_n \text{ with } b = gag^{-1}\}$$

determined by cycle structure $(\nu_1, \ldots, \nu_n)$, index marks length of cycle

Representation theory of the symmetric group S_n

- Constraint $\nu_1 + 2\nu_2 + \cdots + n\nu_n = n$
- Switch variables $\lambda_1 = \nu_1 + \cdots + \nu_n$, $\lambda_2 = \nu_2 + \cdots + \nu_n$, $\ldots$, $\lambda_n = \nu_n$
- Then $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_1 + \cdots + \lambda_n = n$
- $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of n, represented by Young diagram
- Example: $\lambda = (3, 2, 1, 1)$ represented by 
- 1-1 correspondence between conjugacy class and Young diagram

Representation theory of the symmetric group S_n

- Result: # of Young diagrams = # of conjugacy classes = # of inequivalent irreducible representations (irreps) of S_n
- Attach to each distinct Young diagram λ a distinct irrep F_λ of S_n
- Young tabloid corresponding to a Young diagram is a decomposition of $\{1, \ldots, n\}$ into a union of disjoint sets with # of elements given by λ_i. E.g. tabloid for $\lambda = (3, 2, 1, 1)$ is $\{t\} = \{2, 3, 5\}\{1, 7\}\{4\}\{6\}$
- M_λ: set of all tabloids corresponding to λ,

$$\#M_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_n!}.$$

Representation theory of the symmetric group S_n

- Let S_n act on M_λ, get a representation of S_n on $\mathcal{F}(M_\lambda)$, the space of functions on M_λ
- Examples: $\lambda = (n)$, $\lambda = (n-1, 1)$, $\lambda = (n-2, 2)$
- Goal: To each λ corresponds unique "new" irrep F_λ of $\mathcal{F}(M_\lambda)$; the space $\mathcal{F}(M_\lambda)$ decomposes into direct sum of irreps isomorphic to certain of the F_μ with $\mu \geq \lambda$ (with multiplicity) together with the one unique new rep F_λ

$\Rightarrow$ Each Young diagram determines an irrep of S_n.

Representation theory of the symmetric group S_n

- Young tableau t corresponding to λ: assignment of the numbers $1, \ldots, n$ to each of the boxes of λ; order matters! Every Young tableau gives rise to a Young tabloid
- Example: Young diagram $\lambda = (3, 2, 1, 1)$, Young tableau

$$t = \begin{array}{|c|c|c|}\hline 3 & 5 & 2 \\ \hline 1 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \text{ Young tabloid } \{t\} = \{2, 3, 5\}\{1, 7\}\{4\}\{6\}$$

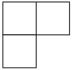
- $C_t =$ subgroup of S_n permuting the numbers in the columns of t among themselves; e.g. for t as above, $C_t = S_{\{3,1,4,6\}} \times S_{\{5,7\}}$

Representation theory of the symmetric group S_n

- $e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{t\}}$ for δ the unit function on $\mathcal{F}(M_\lambda)$
- Then define $F_\lambda = \operatorname{span}(e_t)$ where t ranges over all tableaux corresponding to λ
- Useful: Hook formula

$$\dim F_\lambda = \frac{n!}{\prod_{b \in \lambda} \text{hook length}(b)}$$

- Example: Construct e_t for S_3 with the Young diagram

$\lambda = (2, 1) =$ 

Representation theory of the special unitary group $SU(n)$

Strategy:

- Establish connection between irreps of the symmetric group and irreps of $GL(n)$
- Since $SU(n) \subset SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$, we will get rep of $SU(n)$
- Show that they are irreducible and all of the irreps of $SU(n)$

Representation theory of the special unitary group $SU(n)$

- Look at $V \otimes V$ for V a finite dim. vector space
- S_2 acts on $V \otimes V$ by $(12)x \otimes y = y \otimes x$, has two 1-dim. irreps: trivial rep and sgn rep
- Have decomposition $V \otimes V = S^2(V) \oplus \Lambda^2(V)$ into symmetric and antisymmetric tensors
- $S^2(V)$ is direct sum of $\frac{1}{2}n(n+1)$ copies of trivial rep of S_2
- $\Lambda^2(V)$ is direct sum of $\frac{1}{2}n(n-1)$ copies of sgn rep of S_2
- A linear transformation on V, $A(x \otimes y) = Ax \otimes Ay$, commutes with the action of S_2:

$$A(12)(x\otimes y) = A(y\otimes x) = Ay\otimes Ax = (12)Ax\otimes Ay = (12)A(x\otimes y)$$

- Get rep of $GL(V)$ on $V \otimes V$ that commutes with S_2 (and is irreducible on the subspaces S^2 and Λ^2)

Representation theory of the special unitary group $SU(n)$

- Now decompose $T_r V = V \otimes \cdots \otimes V$ (r factors) into irreducibles
- There exist distinct irreps (ρ_i, U_i) of $GL(V)$, associated with a different rep (σ_i, F_i) of S_r. Have decomposition

$$T_r V = (U_1 \otimes F_1) \oplus \cdots \oplus (U_p \otimes F_p)$$

- $f_i = \dim F_i, s_i = \dim U_i$, then $T_r V$ decomposes under $GL(V)$ into a direct sum of f_i copies of U_i and under S_r into a direct sum of s_i copies of F_i

Representation theory of the special unitary group $SU(n)$

- Take Young diagram λ, let $\dim V = n$. Then define the entries of the dimension table by $d_{ij} = n + j - i$, where i labels the rows and j labels the columns of λ. We have

$$\dim U_\lambda = \frac{\prod_{b\in\lambda} \text{hook length}(b)}{\prod_{b\in\lambda} d(b)}$$

- Example: $\lambda = (4, 3, 1)$

$$d = \begin{array}{cccc} n & n+1 & n+2 & n+3 \\ n-1 & n & n+1 & \\ n-2 & & & \end{array}, \quad \text{hook length} = \begin{array}{|c|c|c|c|} \hline 6 & 4 & 3 & 1 \\ \hline 4 & 2 & 1 & \\ \cline{1-3} 1 & & & \\ \cline{1-1} \end{array}$$

So for $n = 3$, we have

$$\dim U_\lambda = \frac{3\cdot 4\cdot 5\cdot 6\cdot 2\cdot 3\cdot 4}{6\cdot 4\cdot 3\cdot 4\cdot 2} = 15$$

Representation theory of the special unitary group $SU(n)$

- λ a Young diagram, F_λ corresponding irrep of S_r. $T_r V$ has component $W_\lambda = U_\lambda \otimes F_\lambda$, t a tableau of type λ
- Define
$$E_t = \sum_{\sigma \in C_t, \pi \in R_t} \mathrm{sgn}(\sigma)\sigma\pi$$
- Then $E_t(T_r V) = U_\lambda \otimes e_t = \{\phi(e_t) : \phi \in \mathrm{Hom}_{S_r}(F_\lambda, T_r V)\}$, so $E_t(T_r V)$ gives a copy of the irrep U_λ; in particular $E_t(v_{i_1} \otimes \cdots \otimes v_{i_r})$ span the space $E_t(T_r V)$ when $v_1, \dots, v_r$ is a basis of V and the indices i_j range from 1 to n
- Get basis of $E_t(T_r V)$ by arranging the v_{i_j} so that
  - i_j are non-decreasing along the rows
  - strictly increasing on the columns of λ

Representation theory of the special unitary group $SU(n)$

- Goal: the U_λ are all the finite-dim. irreps of $SL(V)$, take $V = \mathbb{C}^n$
- subgroups of $SL(n, \mathbb{C})$

$$N_+ = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}, \quad N_- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\},$$
$$H = \left\{ \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix} : \prod_{i=1}^{n} \delta_i = 1 \right\}$$

- weight vector: simultaneous eigenvector for all elements of H $\Leftrightarrow$ there exists function μ on H s.t. $\rho(\delta)v = \mu(\delta)v$ for all $\delta \in H$
- $\mu(\delta) = \delta_1^{m_1} \cdots \delta_n^{m_n}$, where the weight $m = (m_1, \ldots, m_n)$ is only defined up to adding a constant $s(1, \ldots, 1)$

Representation theory of the special unitary group $SU(n)$

- One can prove: Every finite-dim. rep of $SL(d, \mathbb{C})$ has a maximal weight vector (i.e. a simultaneous eigenvector of $B_+ = H \cdot N_+$), it is determined (up to equivalence) by the corresponding highest weight m
- Passing to the Lie algebra of $SL(d, \mathbb{C})$, one shows that a maximal weight vector must obey

  $$m_1 \geq m_2 \geq \cdots \geq m_n$$

  (set $m_n = 0$ by subtracting $m_n(1, \ldots, 1)$)
- This is just a Young diagram with (at most) $n - 1$ rows!
- In particular, we have constructed all irreps of $SU(n)$

Ordering of energy levels

- Lieb/Mattis (1962), bipartite lattice with Heisenberg Hamiltonian ($N = 2$ in our model); the ground state of H belongs at most to total spin $s := |S_A - S_B|$ and (denoting $E(S)$ the lowest energy eigenvalue with total spin S) $E(S+1) > E(S)$ for all $S \geq s$; $E(S) > E(s)$ for $S > s$ and a special type of lattice
- Proof utilizes M subspaces and the Perron-Frobenius theorem

Ordering of energy levels

- Generalization to our case: space of states $\mathcal{H} = \bigotimes_{i=1}^{|\Lambda|} \mathbb{C}^N$, decompose space as $\mathcal{H} = \bigoplus_\lambda \mathcal{H}_\lambda^N$, λ are Young diagrams with at most $N-1$ rows (since H is invariant under $SU(N)$)
- Denote $E(\lambda)$ relative ground state energy in the space $\mathcal{H}_\lambda^N$
- Equivalent of LM theorem (Hakobyan, 2010 for the chain): If $\lambda \geq \lambda'$, then $E(\lambda') > E(\lambda)$ and the relative ground state energy levels are non-degenerate (inside $\mathcal{H}_\lambda^N$)

Ordering of energy levels

- Perron-Frobenius: If a connected hermitian matrix has no positive off-diagonal elements then its ground state is non-degenerate and has positive components
- Special case of chain: construct non-positive basis

$$|\{x^1\},\ldots,\{x^N\}\rangle = \prod_{\sigma=1}^{N}(c^\dagger_{x_1^\sigma,\sigma}c^\dagger_{x_2^\sigma,\sigma}\cdots c^\dagger_{x_{n_\sigma}^\sigma,\sigma})\,|0\rangle$$

- Basic states in the same subspace $\mathcal{H}^N_\lambda$ are connected by the kinetic term in the Hamiltonian. Obtain uniqueness of relative ground state

$$\Omega_{n_1\ldots n_N} = \sum_{\#\{x^\sigma\}=n_\sigma} \omega_{\{x^1\}\cdots\{x^N\}}\,|\{x^1\},\ldots,\{x^N\}\rangle$$

with strictly positive components from PF theorem