The Heisenberg Ferromagnet

The Heisenberg ferromagnetic spin chain hamiltonian is

$$H_N = -\sum_{n=1}^{N} \vec{S}_n \cdot \vec{S}_{n+1} + \frac{N}{4} \tag{1}$$

with $S_n^j = \sigma_n^j/2$. It commutes with

$$[H_N, S^j] = [H_N, S^{\pm}] = 0 \tag{2}$$

and therefore the eigenvalues are degenerated. The ground state is $\bigotimes_{n=1}^{N} |\uparrow\rangle$.

The Bethe Ansatz

We make the ansatz

$$\psi = \sum_{m_1<\ldots<m_r} a(m_1,\ldots,m_r) S_{m_1}^- \cdots S_{m_r}^- \Psi_0. \tag{3}$$

This leads to the equations

$$Ea(m_1,\ldots,m_r) + \frac{1}{2}\sum_{P\in\mathbb{A}} [a(Pm_1, Pm_2, \ldots, Pm_r) - a(m_1,\ldots,m_r)] = 0 \tag{4}$$

and the equation of the periodic boundary conditions

$$a(m_1,\ldots,m_r) = a(m_2,\ldots,m_r,m_1+N). \tag{5}$$

The $r = 2$ solutions

For two spins pointing down we make an ansatz

$$a(m_1,m_2) = Ae^{im_1k_1+im_2k_2} + A'e^{im_1k_2+im_2k_1}. \tag{6}$$

Writing $A = e^{i\theta/2} = \bar{A}'$ we get the condition

$$2\cot\frac{\theta}{2} = \cot\frac{k_1}{2} - \cot\frac{k_2}{2} \tag{7}$$

and from the periodic boundary conditions

$$Nk_1 - \theta = 2\pi\lambda_1$$
$$Nk_2 + \theta = 2\pi\lambda_2$$

for $\lambda_1, \lambda_2 = 0, \ldots, N-1$. The energy is then

$$E = 2 - \cos(k_1) - \cos(k_2) \tag{8}$$

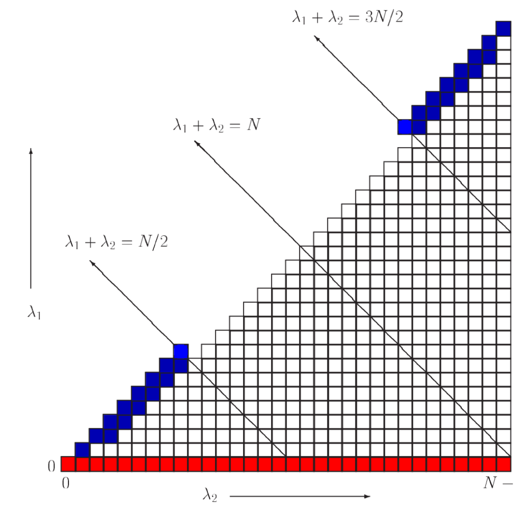

Figure 1: An overview about the solutions for different λ_1, λ_2. In the diagonal one can see the complex bound states. The states in the bottom line are the effective one spin states and in between there are real states. (Karbach and Müller 04)

Real solutions The expression $k = \sum_j k_j = 2\pi\sum_j \lambda_j/N$ is not dependent on θ and therefore k_1 can be determined by solving the equation

$$2\cot\left(\frac{Nk_1}{2}\right) = \cot\left(\frac{k_1}{2}\right) - \cot\left(\frac{k-k_1}{2}\right). \tag{9}$$

Interchanging λ_1, λ_2 gives the same solution so we choose $\lambda_1 \le \lambda_2$. For each $\lambda_1 \le \lambda_2 - 2$ we can find real solutions and there are

$$\sum_{\lambda_2=2}^{N-1} (\lambda_2 - 1) = \binom{N-1}{2} \tag{10}$$

of them.

Potential bound states We use a complex ansatz $k_1 = u + vi = \bar{k}_2$ and get $u = \frac{k}{2}$ and from equation (9)

$$\cos u \sinh(Nv) = \sinh((N-1)v) + \cos(\operatorname{Re}\theta)\sinh v \tag{11}$$

and

$$vi = \theta + \pi(\lambda_1 - \lambda_2). \tag{12}$$

Because $\operatorname{Re}\theta \in \{\pi, 0\}$ we get solutions for

$$\begin{aligned} 0 < \cos u < 1 \qquad & \text{if } \lambda_1 = \lambda_2 \\ 0 < \cos u < 1 - \frac{2}{N} \qquad & \text{if } \lambda_1 = \lambda_2 - 1 \end{aligned}$$

Further $\cos u = 1 \;\Rightarrow\; u = 0 \;\Rightarrow\; v = 0$ is also a valid solution which corresponds to the state $S^- S^- \psi_0$.

Other solutions For the case $\cos u > 1 - \frac{2}{N}$ a new real solution appears in (9).

Further there exists a solution which is formally $u = \pi/2 + i\infty$. This means

$$a(m_1, m_2) = (-1)^{m_1} \delta_{m_1+1, m_2}. \tag{13}$$

Counting the solutions The condition $0 < \cos u \leq 1$ leads to $N-2$ solutions. Together we get

$$\#\text{EV} = \underbrace{\binom{N-1}{2}}_{\text{waves}} + \overbrace{(N-2)}^{\text{possible bound states}} + \underbrace{1}_{u=\pi/2} = \binom{N}{2} \tag{14}$$

which is the desired number of states.

Summary of solutions

$$\begin{aligned} &\lambda_1 = \lambda_2 \;\; \text{bound states} \\ &\lambda_1 = \lambda_2 - 1 \left\langle \begin{array}{ll} \cos u > 1 - \frac{2}{N} & \text{spin waves} \\ \cos u < 1 - \frac{2}{N} & \text{bound states} \end{array} \right. \\ &\lambda_1 < \lambda_2 - 1 \;\; \text{spin waves} \end{aligned}$$

States for small energies

For energies $E \sim N^{-2}$ we get

$$|\lambda_{1,2}| \leq c\sqrt{E}N + 1/2 \tag{15}$$

for a $c > 0$ if we now allow $\lambda_{1,2}$ to take the values $-N/2, \ldots, N/2-1$. For bound states and $E \sim N^{-2}$ we get the estimate

$$v \sim N^{-3/2}. \tag{16}$$

It is possible to show that

$$|E^0_{\lambda_1,\lambda_2} - E_{\lambda_1,\lambda_2}| \leq c(E^{3/2} + N^{-3}) \tag{17}$$

with $c > 0$ and

$$E^0_{\lambda_1,\lambda_2} = \sum_j \left(1 - \cos\left(\frac{2\pi\lambda_j}{N}\right)\right) \tag{18}$$

Bethe Ansatz for $r > 2$

We make the ansatz

$$a(m_1, \ldots, m_r) = \sum_{P \in S_r} \exp\left(i \sum_{j=1}^{r} k_{Pj} m_j + \frac{i}{2} \sum_{i<j} \theta_{Pi,Pj}\right) \tag{19}$$

This leads to the equations

$$\begin{aligned} 2\cot\frac{\theta_{i,j}}{2} &= \cot\frac{k_i}{2} - \cot\frac{k_j}{2} \qquad i \neq j \\ Nk_i - \sum_{j\neq i} \theta_{i,j} &= 2\pi\lambda_i \end{aligned}$$

with $\theta_{i,j} = -\theta_{j,i}$ and $i = 0, \ldots, r$. The Energy is then

$$E = \sum_j \left(2 - \cos(k_j)\right) \tag{20}$$

Rapidities and String states Defining $\Lambda_j = \cot(k_j/2)$ we transform the system to

$$\left(\frac{\Lambda_j + i}{\Lambda_j - i}\right)^N = \prod_{i \neq j}^{r} \frac{\Lambda_j - \Lambda_i + 2i}{\Lambda_j - \Lambda_i - 2i}. \tag{21}$$

For two particles with $\operatorname{Im}\Lambda_j > \varepsilon$ we get in the thermodynamic limit $N \to \infty$ that the left side diverges. To get that the right side diverges we need $|\Lambda_1 - \Lambda_2| \to 2\pi$. A similar statement works for $\operatorname{Im}\Lambda_1 < 0$. In general there exist states

$$\Lambda^{n,j}_\alpha = \Lambda^n_\alpha + i(n + 1 - 2j) + O(\exp(-\delta N))$$

with $j = 1, 2, \ldots, n, \delta > 0$. These are not complete for the Heisenberg magnet but they are complete for a system called the Heisenberg XXZ chain defined by

$$H^\Delta_N = \sum_{j=1}^{N} S^1_n S^1_{n+1} + S^2_n S^2_{n+1} + \Delta S^3_n S^3_{n+1} \tag{22}$$

with $\Delta \neq 1, \Delta > 0$.

The Algebraic Bethe Ansatz

We generalize the Bethe Ansatz and in particular we shall not rely on intelligent guessing for the eigenfunctions. We define the Lax operator on $h_n \otimes k_a$ with $k_a = h_n = \mathbb{C}^2$.

$$L_{n,a}(\lambda) = \lambda I \otimes I + \frac{i}{2}\sum_{\alpha} \sigma^{\alpha} \times \sigma^{\alpha}. \qquad (23)$$

Using the permutation operator $Pa \otimes b = b \otimes a$ defined by $P = (I \otimes I + \sum \sigma^{\alpha} \otimes \sigma^{\alpha})/2$ we can express this by

$$L_{n,a}(\lambda) = \left(\lambda - \frac{i}{2}\right) I_{n,a} + iP_{n,a}. \qquad (24)$$

We define the transport along the chain as

$$T_a(\lambda) = L_{N,a}(\lambda)\cdots L_{1,a}(\lambda) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix}.$$

We can prove the Fundamental Commutation Relation (FCR)

$$R_{a_1,a_2}(\lambda-\mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu) \qquad (25)$$

with $R_{a_1,a_2}(\lambda) = \lambda I_{a_1,a_2} + iP_{a_1,a_2}$. In particular this shows that $[F(\lambda), F(\mu)] = 0$ with

$$F = \operatorname{tr}_a T_a(\lambda) = 2\lambda^N + \sum_{l=0}^{N-2} Q_l \lambda^l \qquad (26)$$

creates a family of $N-1$ commuting operators. In particular

$$H = -\frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln F(\lambda)\Big|_{\lambda=i/2} + \frac{N}{2}. \qquad (27)$$

So we just have to determine the eigenvalues of F.

Commutation Relations The idea is to use B as raising and C as lowering operator to create the eigenstates. We get form the FCR

$$[B(\lambda), B(\mu)] = 0$$
$$A(\lambda)B(\mu) = \frac{\lambda - i}{\lambda} B(\mu)A(\lambda) + \frac{i}{\lambda} B(\lambda)A(\mu)$$
$$D(\lambda)B(\mu) = \frac{\lambda + i}{\lambda} B(\mu)D(\lambda) - \frac{i}{\lambda} B(\lambda)D(\mu)$$

The ground state First we notice that for $\omega_n = |\uparrow\rangle$ we get

$$L_n(\lambda)\begin{pmatrix}\omega_n \\ \omega_n\end{pmatrix} = \begin{pmatrix}\lambda + \frac{i}{2} & * \\ 0 & \lambda - \frac{i}{2}\end{pmatrix}\begin{pmatrix}\omega_n \\ \omega_n\end{pmatrix} \qquad (28)$$

and in particular

$$T(\lambda)\begin{pmatrix}\Omega \\ \Omega\end{pmatrix} = \begin{pmatrix}\left(\lambda + \frac{i}{2}\right)^N & * \\ 0 & \left(\lambda - \frac{i}{2}\right)^N\end{pmatrix}\begin{pmatrix}\Omega \\ \Omega\end{pmatrix}. \qquad (29)$$

We define the ground state as $\Omega \otimes (1,1)$ with $\Omega = \bigotimes_{n=1}^{N} \omega_n$ and we get $C(\lambda)\Omega = 0$. We define the state

$$\Phi(\{\lambda\}) = B(\lambda_1)\cdots B(\lambda_r)\Omega. \qquad (30)$$

With the commutation relations for A, B we get

$$A(\lambda)\Phi(\{\lambda\}) = \prod_{k=1}^{r} \frac{\lambda - \lambda_k - i}{\lambda - \lambda_k}\left(\lambda + \frac{i}{2}\right)^N \Phi(\{\lambda\}) + \sum_{k=1}^{r} N_k(\lambda, \{\lambda\}) B(\lambda_1)\cdots \hat{B}(\lambda_k)\cdots B(\lambda_r)\Omega$$

where $\hat{B}$ means that the term is missing.

We get a similar term for the expression $D(\lambda)\Phi(\{\lambda\})$. If we want the second terms to cancel we get precisely the Bethe Ansatz Equations (21) with $\Lambda = \lambda/2$. The eigenvalues $F(\lambda)\Phi(\{\lambda\}) = \Lambda(\lambda, \{\lambda\})\Phi(\{\lambda\})$ take the form

$$\Lambda(\lambda, \{\lambda\}) = \left(\lambda + \frac{i}{2}\right)^N \prod_{j=1}^{r} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} + \left(\lambda - \frac{i}{2}\right)^N \prod_{j=1}^{r} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j}$$

In particular we see that the second product vanishes for $\lambda = i/2$. This ensures that the energies get their additive form.

References

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