An analogue of the Lieb-Thirring inequality for positive temperature in $d = 1$

Johannes Alt

In 2011, Frank, Lewin, Lieb and Seiringer gave lower bounds on the energy difference of the Fermi sea in all dimensions d which is caused by adding a one-body potential. However, for $d = 1$ the estimate contains an additional term which is not present in higher dimensions. For positive temperature, we provide a bound on the additional term for $d = 1$ in terms of two L^p norms of the potential.

1 Introduction

In this note we are interested in the change of the energy of a Fermi sea when adding a one-body potential V to $-\Delta - \lambda$. At zero temperature this energy difference is described by $\operatorname{tr}_V(-\Delta - \lambda + V)Q_{\lambda,V}$. (We freely use the notation introduced in [1].)

For $d \geq 2$ the following analogue of the Lieb-Thirring inequality are proved in [1]

$$\operatorname{tr}_V(-\Delta - \lambda + V)Q_{\lambda,V} \geq -L(d)\int_{\mathbb{R}^d} (V(x)-\lambda)_-^{1+\frac{d}{2}} - \lambda_+^{1+\frac{d}{2}} + \frac{2+d}{2}\lambda_+^{\frac{d}{2}} V(x)\mathrm{d}x \tag{1.1}$$

for real-valued $V \in L^2(\mathbb{R}^d) \cap L^{1+\frac{d}{2}}(\mathbb{R}^d)$ where $Q_{\lambda,V} := \Pi_V^- - \Pi^-$ with $\Pi_V^- := \mathbb{1}(-\Delta - \lambda + V \leq 0)$ and $\Pi_- := \mathbb{1}(-\Delta - \lambda \leq 0)$ the orthogonal projections onto the negative parts of the corresponding (self-adjoint) operators.

On the other hand, for $d = 1$ they get the weaker estimates

$$\begin{aligned}\operatorname{tr}_V(-\Delta - \lambda + V)Q_V \geq& -L(1)\int_{\mathbb{R}} (V(x)-\lambda)_-^{3/2} - \lambda^{3/2} + \frac{3}{2}\lambda^{1/2}V(x)\mathrm{d}x \\ & -L'(1)\int_{\mathbb{R}} \frac{\sqrt{\lambda}+|k|}{\sqrt{\lambda}|k|}\log\left(\frac{2\sqrt{\lambda}+|k|}{|2\sqrt{\lambda}-|k||}\right)|\hat{V}(k)|^2\mathrm{d}k \end{aligned}\tag{1.2}$$

for $\lambda > 0$ and

$$\operatorname{tr}_V(-\Delta - \lambda + V)Q_V \geq -L(1)\int_{\mathbb{R}} (V(x)-\lambda)_-^{3/2}\mathrm{d}x$$

for $\lambda \leq 0$. Note that (1.2) has an additional term compared to (1.1). We will see in section 2 that this additional term diverges logarithmically for $\lambda \downarrow 0$.

We are interested in the Fermi gas at a positive temperature T and a chemical potential μ. In this case we have to consider the Fermi-Dirac distribution for the free energy

$$f_{T,\mu}(\lambda) = -T\log\left(1 + \mathrm{e}^{-(\lambda-\mu)/T}\right)$$

with the second derivative

$$f''_{T,\mu}(\lambda) = -\frac{\mathrm{e}^{-(\lambda-\mu)/T}}{T\left(1+\mathrm{e}^{-(\lambda-\mu)/T}\right)^2}$$

for $T, \mu > 0$. (We set $f := f_{T,\mu}$.) Then the difference in the free energy caused by adding a one-body potential V is given by

$$\int_{\mathbb{R}} \operatorname{tr}_V(-\Delta + V - \lambda)Q_{\lambda,V} f''(\lambda)\mathrm{d}\lambda. \tag{1.3}$$

Our main goal is to study the additional term in (1.2) for positive temperature T and a chemical potential μ i.e. the integral

$$I_1(f,V) := L'(1)\int_0^\infty f''_{T,\mu}(\lambda)\int_{\mathbb{R}} \frac{\lambda^{1/2}+|k|}{\lambda^{1/2}|k|}\log\left(\frac{2\lambda^{1/2}+|k|}{|2\lambda^{1/2}-|k||}\right)|\hat{V}(k)|^2 \mathrm{d}k\mathrm{d}\lambda. \tag{1.4}$$

Note that we can restrict the k-integration in (1.4) from $\mathbb{R}$ to $[0,\infty)$ by adding a factor 2 since $\hat{V}(-k) = \overline{\hat{V}(k)}$ for all $k \in \mathbb{R}$ as V is real-valued.

2 Divergence of the additional term in (1.2) for $\lambda \downarrow 0$

For $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\lambda > 0$ we define

$$I(V,\lambda) := \int_0^\infty \frac{\sqrt{\lambda}+k}{\sqrt{\lambda}k}\log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right)|\hat{V}(k)|^2\mathrm{d}k.$$

By Fatou's lemma we get

$$\liminf_{\lambda\downarrow 0} I(V,\lambda) \geq \int_0^\infty \frac{1+y}{y}\log\left(\frac{2+y}{|2-y|}\right)\lim_{\lambda\downarrow 0}|\hat{V}(\sqrt{\lambda}y)|^2\mathrm{d}y = |\hat{V}(0)|^2\int_0^\infty \frac{1+y}{y}\log\left(\frac{2+y}{|2-y|}\right)\mathrm{d}y = \infty$$

where we used the substitution $k = \sqrt{\lambda}y$ in the second step and $\hat{V} \in C(\mathbb{R})$ in the third step.

Next, we analyze the divergence rate of $I(V,\lambda)$ for $\lambda \downarrow 0$.

Lemma 2.1. *We have*

$$\lim_{\lambda\downarrow 0}\frac{I(V,\lambda)}{\log(1/\sqrt{\lambda})} = 4|\hat{V}(0)|^2.$$

Proof. We define

$$L(V) := \lim_{\lambda\downarrow 0}\frac{I(V,\lambda)}{\log(1/\sqrt{\lambda})}.$$

For every $\varepsilon > 0$ we get using the substitution $k = \sqrt{\lambda}y$

$$\begin{aligned}
\frac{1}{\log(1/\sqrt{\lambda})}\int_\varepsilon^\infty \frac{\sqrt{\lambda}+k}{\sqrt{\lambda}k}\log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right)|\hat{V}(k)|^2\mathrm{d}k &= \frac{1}{\log(1/\sqrt{\lambda})}\int_{\varepsilon/\sqrt{\lambda}}^\infty \frac{1+y}{y}\log\left(\frac{2+y}{|2-y|}\right)|\hat{V}(\sqrt{\lambda}y)|^2\mathrm{d}y \\
&\leq \frac{C_\varepsilon}{\log(1/\sqrt{\lambda})}\int_{\varepsilon/\sqrt{\lambda}}^\infty |\hat{V}(\sqrt{\lambda}y)|^2\frac{\mathrm{d}y}{y} \\
&= \frac{C_\varepsilon}{\log(1/\sqrt{\lambda})}\int_\varepsilon^\infty |\hat{V}(y)|^2\frac{\mathrm{d}y}{y} \\
&\xrightarrow{\lambda\downarrow 0} 0.
\end{aligned}$$

Therefore, we can replace the upper bound of integration in the definition of L by any $\varepsilon > 0$ i.e. we redefine

$$L(V) := \lim_{\lambda\downarrow 0}\frac{1}{\log(1/\sqrt{\lambda})}\int_0^\varepsilon \frac{\sqrt{\lambda}+k}{\sqrt{\lambda}k}\log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right)|\hat{V}(k)|^2\mathrm{d}k = \lim_{\lambda\downarrow 0}\frac{I(V,\lambda)}{\log(1/\sqrt{\lambda})}.$$

Since $\hat{V} \in C(\mathbb{R})$ (as $V \in L^1(\mathbb{R})$) we find for every $\delta > 0$ an $\varepsilon > 0$ such that

$$|\hat{V}(0)|^2 - \delta \leq |\hat{V}(k)|^2 \leq |\hat{V}(0)|^2 + \delta$$

for all $k \in [0,\varepsilon]$.

Moreover, we have

$$x - R(x) \leq \log(1+x) \leq x + R(x) \tag{2.1}$$

with $R(x) = |x|^2/(2(1-|x|))$ for $|x|<1$. This implies

$$\int_0^\varepsilon \frac{k+\sqrt{\lambda}}{k\sqrt{\lambda}} \log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right) \mathrm{d}k \geq \int_{10}^{\varepsilon/\sqrt{\lambda}} \log\left(1+\frac{4}{y-2}\right) \mathrm{d}y \geq 4\log\left(\frac{\varepsilon}{\sqrt{\lambda}}-2\right) - C.$$

Therefore, we get

$$4(|\hat{V}(0)|^2-\delta) \leq \lim_{\lambda\downarrow 0} \frac{1}{\log(1/\sqrt{\lambda})} \int_0^\varepsilon \frac{\sqrt{\lambda}+k}{\sqrt{\lambda}k} \log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right) |\hat{V}(k)|^2 \mathrm{d}k.$$

As $\delta > 0$ was arbitrary we get $L(V) \geq 4|\hat{V}(0)|^2$. Using (2.1) we get for every sufficiently large L

$$\begin{aligned}\int_0^\varepsilon \frac{k+\sqrt{\lambda}}{k\sqrt{\lambda}} \log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right) \mathrm{d}k &= \int_0^L \frac{1+y}{y}\log\left(\frac{2+y}{|y-2|}\right)\mathrm{d}y + \int_L^{\varepsilon/\sqrt{\lambda}} \frac{1+y}{y}\log\left(1+\frac{4}{y-2}\right)\mathrm{d}y \\ &\leq C_L + 4\frac{L+1}{L}\log\left(\frac{\varepsilon}{\sqrt{\lambda}}-2\right).\end{aligned}$$

Thus, we have $L(V) \leq \frac{4(L+1)}{L}(|\hat{V}(0)|^2+\delta)$ and therefore, as δ and L were arbitrary we get $L(V) \leq 4|\hat{V}(0)|^2$. □

3 Estimate on the additional term in (1.2)

Next, we want to analyze the function $G(k) := \int_0^\infty g(k,\lambda)\mathrm{d}\lambda$ where

$$g(k,\lambda) := -\frac{\mathrm{e}^{-(\lambda-\mu)/T}}{T(1+\mathrm{e}^{-(\lambda-\mu)/T})^2} \frac{\sqrt{\lambda}+k}{k\sqrt{\lambda}} \log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right).$$

In particular, we want to analyze which L^p spaces contain G. Our interest is caused by the fact that Fubini's Theorem allows us to rewrite

$$I_1(f,V) = 2L'(1)\int_0^\infty G(k)|\hat{V}(k)|^2\mathrm{d}k. \tag{3.1}$$

Using Fatou's lemma and the limits

$$\begin{aligned}\lim_{k\downarrow 0} g(k,\lambda) &= -\frac{1}{T\sqrt{\lambda}}\frac{\mathrm{e}^{-(\lambda-\mu)/T}}{(1+\mathrm{e}^{-(\lambda-\mu)/T})^2}, \\ \lim_{k\to\infty}\frac{g(k,\lambda)}{k^{-1}} &= -\frac{4}{T}\frac{\mathrm{e}^{-(\lambda-\mu)/T}}{(1+\mathrm{e}^{-(\lambda-\mu)/T})^2}\end{aligned}$$

which follow from applying l'Hôpital's rule we get the following (finite) lower bounds for the behaviour of G for $k\to 0$ and $k\to\infty$

$$\begin{aligned}\liminf_{k\downarrow 0}|G(k)| &\geq \int_0^\infty \frac{1}{T\sqrt{\lambda}}\frac{\mathrm{e}^{-(\lambda-\mu)/T}}{(1+\mathrm{e}^{-(\lambda-\mu)/T})^2}\mathrm{d}\lambda, \\ \liminf_{k\to\infty} k|G(k)| &\geq \int_0^\infty \frac{4}{T}\frac{\mathrm{e}^{-(\lambda-\mu)/T}}{(1+\mathrm{e}^{-(\lambda-\mu)/T})^2}\mathrm{d}\lambda = \frac{4}{1+\mathrm{e}^{\mu/T}}.\end{aligned}$$

Using the estimate $f''(\lambda) \leq \exp(-|\lambda-\mu|/T)$ and the substitution $\lambda = k^2\lambda'$ we get

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-(\lambda-\mu)/T}}{T(1+\mathrm{e}^{-(\lambda-\mu)/T})^2}\frac{k+\sqrt{\lambda}}{k\sqrt{\lambda}}\log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right)\mathrm{d}\lambda &\leq \int_0^\infty \frac{\mathrm{e}^{-|\lambda-\mu|/T}}{T}\frac{k+\sqrt{\lambda}}{k\sqrt{\lambda}}\log\left(\frac{2\sqrt{\lambda}+k}{|2\sqrt{\lambda}-k|}\right)\mathrm{d}\lambda \\ &= \int_0^\infty \frac{k}{T}\mathrm{e}^{-k^2|\lambda-\mu/k^2|/T}\frac{1+\sqrt{\lambda}}{\sqrt{\lambda}}\log\left(\frac{2\sqrt{\lambda}+1}{|2\sqrt{\lambda}-1|}\right)\mathrm{d}\lambda.\end{aligned}$$

Our goal is to prove that G is contained in $L^{3/2}$. Therefore, we distinguish between small k and large k and define first the following auxiliary functions

$$
\begin{aligned}
h_1(k,\lambda) &:= \frac{k}{T}\mathrm{e}^{-k^2|\lambda-\mu/k^2|/T},\\
h_2(\lambda) &:= \frac{1+\sqrt{\lambda}}{\sqrt{\lambda}}\log\left(\frac{2\sqrt{\lambda}+1}{|2\sqrt{\lambda}-1|}\right).
\end{aligned}
$$

Thus,

$$
|G(k)| \le \int_0^\infty h_1(k,\lambda)h_2(\lambda)\mathrm{d}\lambda
$$

and we can use Hölder's inequality to estimate the integral on the right hand side.

Note that h_2 has for $\lambda = 0$ the following behaviour

$$
\lim_{\lambda\downarrow 0}\left(1+\frac{1}{\sqrt{\lambda}}\right)\log\left(\frac{2\sqrt{\lambda}+1}{|2\sqrt{\lambda}-1|}\right) = \lim_{\lambda\downarrow 0}\frac{4}{(1+2\sqrt{\lambda})(1-2\sqrt{\lambda})} = 4
$$

where we used l'Hôpital's rule in the first step. The logarithmic singularity at $\lambda = 1/4$ lies in L^p for all $p \ge 1$ since

$$
\lim_{x\downarrow 0} x^\alpha \log(x)^\beta = 0
$$

for all $\alpha, \beta > 0$. For large λ the function h_2 behaves like $\lambda^{-1/2}$. Thus, $h_2 \in L^p[0,\infty)$ for all $p > 2$.

On the other hand, we have

$$
\begin{aligned}
\|h_1(k,\cdot)\|_p^p &= \frac{k^p}{T^p}\int_0^\infty \mathrm{e}^{-pk^2|\lambda-\mu/k^2|/T}\mathrm{d}\lambda\\
&= \frac{k^p}{T^p}\left(\left[\frac{T}{pk^2}\mathrm{e}^{pk^2\lambda/T}\right]_{-\mu/k^2}^{0} + \left[-\frac{T}{pk^2}\mathrm{e}^{-pk^2\lambda/T}\right]_0^\infty\right)\\
&= \frac{k^{p-2}}{pT^{p-1}}\left(2-e^{-p\mu/T}\right) \le 2\frac{k^{p-2}}{pT^{p-1}},
\end{aligned}
$$

which implies

$$
|G(k)| \le \left(\frac{2}{p}\right)^{1/p}\frac{k^{1-2/p}}{T^{1-1/p}} \tag{3.2}
$$

for $p \in (1,2)$ by Hölder's inequality. Note that this constant diverges for $T \to 0$ and that the (possible) divergence of G at zero lies in every L^p space.

For $k^2 > T+1$ we define the auxiliary functions

$$
\begin{aligned}
h_3(k,\lambda) &:= \frac{k}{T}\mathrm{e}^{-k^2|\lambda-\mu/k^2|/T}\mathrm{e}^{\lambda},\\
h_4(\lambda) &:= \mathrm{e}^{-\lambda}\frac{1+\sqrt{\lambda}}{\sqrt{\lambda}}\log\left(\frac{2\sqrt{\lambda}+1}{|2\sqrt{\lambda}-1|}\right).
\end{aligned}
$$

As before,

$$
|G(k)| \le \int_0^\infty h_3(k,\lambda)h_4(\lambda)\mathrm{d}\lambda
$$

which can be estimated by Hölder's inequality. Using the previous considerations and fact that h_4 decays exponentially for $\lambda \to \infty$ we get $h_4 \in L^q[0,\infty)$ for $1 < q < \infty$. For $p > 1$ and $k^2 > T+1$ we have

$$\begin{aligned}\|h_3(k,\cdot)\|_p^p &= \frac{k^p}{T^p}\left(\int_0^{\mu/k^2} \mathrm{e}^{-p\mu/T+p(k^2/T+1)\lambda}\mathrm{d}\lambda + \int_{\mu/k^2}^{\infty} \mathrm{e}^{p\mu/T-p(k^2/T-1)\lambda}\mathrm{d}\lambda\right)\\ &= \frac{k^p}{T^p}\left(\mathrm{e}^{-p\mu/T}\left[\frac{T}{p(k^2+T)}\mathrm{e}^{p(k^2/T+1)\lambda}\right]_0^{\mu/k^2} + \mathrm{e}^{p\mu/T}\left[-\frac{T}{p(k^2-T)}\mathrm{e}^{-p(k^2/T-1)\lambda}\right]_{\mu/k^2}^{\infty}\right)\\ &= \frac{k^p}{pT^{p-1}}\left(\frac{\mathrm{e}^{\mu p/k^2}-\mathrm{e}^{-p\mu/T}}{k^2+T} + \frac{\mathrm{e}^{\mu p/k^2}}{k^2-T}\right).\end{aligned}$$

Thus, we have for $k^2 > T+1$ the estimate

$$|G(k)| \le Ck^{1-2/p} \tag{3.3}$$

for $p > 1$ by Hölder's inequality.

Combining the estimates in (3.2) and (3.3) we get $G \in L^p[0,\infty)$ for $1 < p < \infty$. Thus, in particular $G \in L^{3/2}[0,\infty)$ and we have by Hölder's inequality

$$\int_0^\infty G(k)|\hat{V}(k)|^2\mathrm{d}k \ge \|\hat{V}\|_\infty \int_0^\infty G(k)|\hat{V}(k)|\mathrm{d}k \ge -\|V\|_1\|G\|_{3/2}\|\hat{V}\|_3 \ge -C\|G\|_{3/2}\|V\|_1\|V\|_{3/2}$$

where C is the operator norm of the Fourier transform $L^{3/2} \to L^3$. (Note $G(k) \le 0$ for all k.) This implies that (3.1) is finite for all $V \in L^1(\mathbb{R}) \cap L^{3/2}(\mathbb{R})$.

Similarly, Hölder's inequality for $p \in (1,2)$ yields

$$\int_0^\infty G(k)|\hat{V}(k)|^2\mathrm{d}k \ge \|G\|_{\frac{p}{2-p}}\|\hat{V}\|^2_{\frac{p-1}{p}} \ge C\|G\|_{\frac{p}{2-p}}\|V\|_p^2$$

where C equals the squared operator norm of the Fourier transform $L^p \to L^{(p-1)/p}$.

References

[1] R. L. Frank, M. Lewin, E. H. Lieb, R. Seiringer: *A positive density analogue of the Lieb–Thirring inequality. Duke Math. J.*, **162**(2013), no. 3, 435–495. URL `http://arxiv.org/abs/1108.4246`.