Chapter 1

Random Matching Problem

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Introduction

Suppose to have two sets of points in $\mathbb{R}^d$ given by $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$; the *matching problem*, also known as *assignment problem*, deal with the study of matching bijectively the first set of points $(x_i)_i$ with the second one $(y_j)_j$ in such a way to minimize an average cost (usually supposed to be the distance between the points), that is:

$$\min_{\sigma \in \Sigma_n} \left\{ \frac{1}{n} \sum_{i=1}^n |x_i - y_{\sigma(i)}|^p \right\}$$

where $1 \le p < +\infty$.

We speak about *random matching problem* when we don't work with deterministic points but indeed with random variables. Precisely, let $(X_1, ..., X_n), (Y_1, ..., Y_n)$ be independent and identically distributed (IID) random variables with law $\mu \in \mathscr{P}_p(\Omega)$, with $\Omega \subset \mathbb{R}^n$; we can then consider the sequence of random variables obtained as the optimal matching cost between any realizations of X_i and Y_j, namely:

$$\mathcal{M}_n := \min_{\sigma \in \Sigma_n} \left\{ \frac{1}{n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^p \right\}. \tag{1.1}$$

The main goal of the theory is to study the asymptotic behaviour of the optimal cost (or its average behaviour, in the case of the random formulation of the problem).

Remark 1. Let us recall the concept of Wasserstein distance in the space of probability measures. Let $\mu, \nu \in \mathscr{P}_p(\Omega)$; we define the *$p-$Wasserstein distance* between μ and ν as:

$$W_p^p(\mu, \nu) = \min_{\pi} \left\{ \int_{\Omega \times \Omega} |x - y|^p \, \mathrm{d}\pi(x, y) \;:\; (\mathcal{P}_1)_{\#}(\pi) = \mu, \; (\mathcal{P}_2)_{\#}(\pi) = \nu \right\}$$

where $\pi \in \mathscr{P}(\Omega \times \Omega)$ (called transport plan), whereas $\mathcal{P}_i$ is the projections on the $i - th$ factor of the product space $\Omega \times \Omega$.

The random matching optimal cost can be reformulated in terms of the Wasserstein distance; let's consider the following measures (called *empirical measures*):

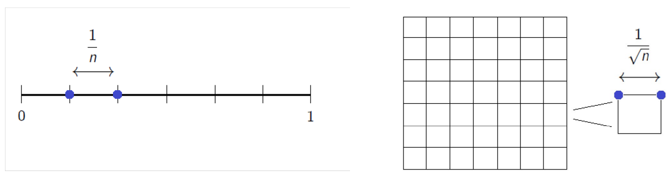
$$\mu_n^X := \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \quad \mu_n^Y := \frac{1}{n} \sum_{j=1}^n \delta_{Y_j}; \tag{1.2}$$

then computing the Wasserstein distance between such measures we get:

$$W_p^p\left(\frac{1}{n}\sum_{i=1}^n \delta_{X_i}, \sum_{j=1}^n \delta_{Y_j}\right) = \min\left\{\int_\Omega |x - T(x)|^p \,\mathrm{d}\mu_n^X \;:\; T_\# \mu_n^X = \mu_n^Y\right\}$$
$$= \min_{\sigma\in\Sigma_n}\left\{\frac{1}{n}\sum_{i=1}^n |X_i - Y_{\sigma(i)}|^p\right\} = \mathcal{M}_n.$$

1.1 Naive approach and known results

Supposing to work with a uniform measure μ on the $d-$dimensional cube $[0,1]^d$; let's take n points and suppose them to be equidistributed on the cube. Then the natural distance between two close points after n steps would be expected to be of the order $n^{-\frac{1}{d}}$, that means we expect $\mathcal{M}_n$ to be of the order $n^{-\frac{p}{d}}$.



This simple and basic guess turns out to be correct in dimension $d \geq 3$; indeed Dobríc and Yukic [5] proved:

$$\frac{1}{C}\, n^{-\frac{p}{d}} \leq \mathbb{E}(\mathcal{M}_n) \leq \; Cn^{-\frac{p}{d}}$$

where the law μ is supposed to be the uniform on the $d-$dimension cube. Surprisingly, in dimension $d = 2$ we find a deviation from the ansatz for a logarithmic factor; precisely, Atjas-Komlós-Tusnády in [1] proved[1]:

$$\frac{1}{C}\left(\frac{logn}{n}\right)^{\frac{p}{2}} \leq \mathbb{E}(\mathcal{M}_n) \leq C\left(\frac{logn}{n}\right)^{\frac{p}{2}}.$$

Nevertheless, the previous results are actually known in this generality only for the uniform case; if we work with different measures, especially with non compactly supported ones, the asymptotic behaviour can be different.

Under some special combinatorial assumptions between p and d, some other polynomial behaviours have been proved in a more general setting, not only limited to the uniform case (even for some non compactly supported measures).

A really deep result has been proved by Ambrosio, Stra and Trevisan [2] in 2006, concerning the setting of a $2-$Riemannian compact manifold.

Theorem 2 (Ambrosio, Stra, Trevisan)**.** *Let (M, g) be a $2-$ dimensional compact manifold and let $\mu = \frac{1}{vol_g(M)} vol_g \in \mathscr{P}_2(M)$ be the (renormalized) uniform measure on M, where vol_g is the canonical volume measure on (M, G) Let $(X_1, ..., X_n)$, $(Y_1, ..., Y_n)$ be IID random variables with law μ. Then we have:*

$$\lim_{n\to+\infty} \frac{n}{logn} W_2^2(\mu_n^X, \mu_n^Y) = \frac{1}{2\pi}, \tag{1.3}$$

$$\lim_{n\to+\infty} \frac{n}{logn} W_2^2(\mu_n^X, \mu) = \frac{1}{4\pi}. \tag{1.4}$$

[1] They actually proved only the case $p = 1$, but it was done afterwards.

Surprisingly, the previous result is not depending on the space we are working with; in fact, the infinitesimal behaviour of the minimal cost is strictly related to the small time behaviour of the kernel of the associated heat semigroup.

Generally speaking, given a measure $\mu = e^{-V}\,\mathrm{d}vol_g$, one can consider the second-order differential operator L defined by:

$$L : \mathrm{D}(L) \subset L^2(M,\mu) \to L^2(M,\mu)$$

$$\int_M \varphi(-L\psi)\,\mathrm{d}\mu = \int_M \langle \nabla\varphi, \nabla\psi\rangle\,\mathrm{d}\mu, \quad \forall \varphi, \psi \in C_c^\infty(M). \tag{1.5}$$

In particular, it can be equivalently defined in the differential form:

$$L = \Delta_g - \nabla V \cdot \nabla \tag{1.6}$$

where Δ_g is the Laplace-Beltrami operator on (M,g). Denote by P_t the semigroup with generator L and with p_t the correspondent kernel, that is:

$$\dot{P}_t = LP_t$$

and

$$P_t\varphi(x) = \int_M p_t(x,y)\varphi(y)\,\mathrm{d}\mu(y), \quad x \in M,\ \varphi \in C^\infty(M),\ t > 0.$$

One has $\int_M p_t(x,y)\,\mathrm{d}\mu(y) = 1$ for any $t > 0$ and $x \in M$; moreover, as we will see in the next section, the minimal matching problem can be associated with the correspondent semigroup; in particular, the limit in [AST] is strictly related to the small time behaviour of p_t, namely:

$$\lim_{t\to 0} 4\pi t \int_M p_t(x,x)\,\mathrm{d}\mu(x) = 1 \tag{1.7}$$

where the previous limit holds for the renormalized uniform measure on a $2-$compact Riemannian manifold.

Another important property we will use again in future is the so-called *Chapman-Kolmorogov* equation; it can be formulated as follows:

$$p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)\,\mathrm{d}\mu(z), \quad \forall t,s > 0,\ \forall x,y,z \in M \tag{1.8}$$

and can be interpreted in a probabilistic point of view as the Markov property of the stochastic process that has p_s as transition probability function in time $s > 0$.

1.2 A PDE approach to random matching problems

From now on, we will focus on the so-called *monopartite problem*, namely:

$$M_n := W_2^2(\mu_n, \mu) \tag{1.9}$$

where as before $\mu \in \mathscr{P}(\mathbb{R}^n)$ and

$$\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i} \tag{1.10}$$

with X_i IID random variables with law μ. The monopartite problem behaviour gives us always an upper bound on the original problem (up to a factor two, using the triangle inequality) but in fact it turns out to be equivalent in the limit of $n \to +\infty$ (*Comment*: is it always right?).

The technique we are going to introduce here is mainly due to M. Ledoux, [6], following a similar PDE-approach to the one of [2] but it can be applied in a more general setting. We are not interested by now in the computation of an exact limit (as in [2]) but we look for a simpler asymptotic behaviour (mostly concerning the upper bounds) of some of the cases that were not completely understood or proved before Ledoux's work.

In particular, one of the main new result proved in [6] are the following bounds concerning the Gaussian samples in dimension 2 with $p = 2$:

$$\frac{1}{C}\frac{logn}{n} \le \mathbb{E}\left[W_2^2(\mu_n, \mu)\right] \le C\frac{(logn)^2}{n} \tag{1.11}$$

where μ is the Gaussian measure in $\mathbb{R}^2$.

Nevertheless, as already pointed out, the approach in [L] is not restricted to the gaussian case in dimension 2 and in the next section we will see the main ideas behind it.

1.2.1 Regularization of empirical measures

Let's suppose to have $\mu = e^{-V} vol_g \in \mathscr{P}(M)$ with M a $d-$dimensional Riemannian manifold and $V : M \to \mathbb{R}$ a smooth potential; as before, denote by L the generator of the semigroup associated with μ.

The main idea is to regularize the empirical measure μ_n with another measure $\tilde{\mu}_n << \mu$ absolutely continuous with respect to μ in such a way to have $W_2^2(\mu_n, \tilde{\mu}_n)$ small enough and at the same time be able to control the Wasserstein distance between $\tilde{\mu}_n$ and μ. In fact, given an absolutely continuos measure ν with respect to μ, the key observation is given by the following lemma (Ledoux, [6]).

Proposition 3. *Let $\mu \in \mathscr{P}(M)$ and $\nu = f\mu \in \mathscr{P}(M)$. Then we have:*

$$W_p(\mu, \nu) \le p\left\| f - 1 \right\|_{H_\mu^{-1,p}(M)} \tag{1.12}$$

where $\|\cdot\|_{H_\mu^{-1,p}(M)}$ denotes the dual norm in $H_\mu^p(M)$, namely:

$$\|g\|_{H_\mu^{-1}(M)} = \sup\left\{\int_M g\varphi \,\mathrm{d}\mu \;:\; \int_M |\nabla\varphi|^p \,\mathrm{d}\mu \le 1\right\}.$$

Remark 4. It's not difficult to prove, from the definition of L, the following formula for the dual norm in $H_\mu^p(M)$:

$$\|g\|_{H_\mu^{-1}(M)}^p = \int_M |\nabla\left((-L)^{-1}g\right)|^p \,\mathrm{d}\mu. \tag{1.13}$$

In particular, the functions with finite dual norm are the ones in the domains of the inverse of the operator L. Moreover, we have a nice representation of such an inverse operator integrating the semigroup on the whole real line:

$$(-L)^{-1} = \int_0^{+\infty} P_t \,\mathrm{d}t. \tag{1.14}$$

Using the previous formula, in the special case of $p = 2$, it is possible to write the dual norm of a function using a simpler description:

$$\begin{aligned}\int_M |\nabla \left((-L)^{-1} g\right)|^p \,\mathrm{d}\mu &= \int_M g(-L)^{-1} g \,\mathrm{d}\mu \\ &= \int_0^{+\infty} \int_M g P_t g \,\mathrm{d}\mu \,\mathrm{d}t \qquad (1.15)\\ &= 2\int_0^{+\infty} \int_M (P_t g)^2 \,\mathrm{d}\mu \,\mathrm{d}t\end{aligned}$$

where in the last equality we used the symmetry of the semigroup:

$$\int_M g P_t f \,\mathrm{d}\mu = \int_M f P_t g \,\mathrm{d}\mu, \quad \forall f, g : M \to \mathbb{R}.$$

Remark 5. Obviously one could wonder how much rough such an estimate could be; in fact, the right-hand side of the latter bound turns out to be the correct infinitesimal order of the correspondent Wasserstein distance, as one can see in the following result.

Lemma 6. *Let $g \in L^2(\mu)$ be a zero-mean function, that is $\int g \,\mathrm{d}\mu = 0$; let us define $\mu_\epsilon := (1 + \epsilon g)\mu$, for any $\epsilon > 0$. Then:*

$$\lim_{\epsilon \to 0} \frac{W_2^2(\mu, \mu_\epsilon)}{\epsilon^2} = \|g\|^2_{H_\mu^{-1}(M)}. \qquad (1.16)$$

We give a possible proof of the previous result inside the following chapter (Lemma 11).

Having these results in mind, the main goal is to find a suitable regularization for the empirical measure; the idea is to consider the kernel of the semigroup P_t given by p_t and to define:

$$\mathrm{d}\mu_{n,t} := \frac{1}{n} \sum_{i=1}^n p_t(X_i, \cdot) \,\mathrm{d}\mu, \quad t > 0. \qquad (1.17)$$

First of all, we need to estimate the error we are making using this approximation; using convexity of Wasserstein distance, we obtain:

$$W_p^p(\mu_{n,t}, \mu_n) \le \frac{1}{n} \sum_{i=1}^n W_p^p(\delta_{X_i}, p_t(X_i, \cdot)\mu) = \frac{1}{n} \sum_{i=1}^n \int_M p_t(X_i, y)|X_i - y|^p \,\mathrm{d}\mu(y); \qquad (1.18)$$

therefore, using that the random variables are IID, we get:

$$\mathbb{E}\left[W_p^p(\mu_{n,t}, \mu_n)\right] \le \int_M \int_M p_t(x, y)|x - y|^p \,\mathrm{d}\mu(y) \,\mathrm{d}\mu(x), \quad \forall n \in \mathbb{N}. \qquad (1.19)$$

Recall that the kernel of a semigroup can be equivalently defined as the solution of the flow of L with a delta as initial condition, precisely:

$$p_t(x, y) \text{ is the solution of } \dot{p}_t = L p_t, \quad p_0(x, y) = \delta_x(y).$$

Therefore, Equation 1.19 easily yields:

$$\mathbb{E}\left[W_p^p(\mu_{n,t}, \mu_n)\right] \to 0, \quad t \to 0$$

giving us the sought approximation. We'll see afterwards how to obtain a more specific result concerning such a convergence. Before doing that, let's have a look to the other part of the cost, namely the one between the approximated empirical measure and μ:

$$\mathbb{E}\left[W_p^p(\mu_{n,t}, \mu)\right].$$

For simplicity, we will limit our analysis to the case of $p = 2$; a more general proof can be done even for different choices for $p \geq 1$ but in a slightly different way, mainly because of the particular symmetry of the problem in dimension 2.

Therefore assuming this, by applying Proposition 3 with $f = f_{n,t} = \frac{1}{n}\sum_{i=1}^{n} p_t(X_i, \cdot)$, we obtain:

$$\mathbb{E}\left[W_2^2(\mu_{n,t}, \mu)\right] \leq 4\mathbb{E}\,\|f_{n,t} - 1\|^2_{H_\mu^{-1}(M)}\,. \tag{1.20}$$

Using the integral representation as in Equation 1.14, in fact we obtain:

$$\|f_{n,t} - 1\|^2_{H_\mu^{-1}(M)} = 2\int_0^\infty \int_M [P_s(f_{n,t}-1)]^2 \,\mathrm{d}\mu\,\mathrm{d}s = 2\int_t^\infty \int_M \left[\frac{1}{n}\sum_{i=1}^{n}[p_s(X_i, y) - 1]\right]^2 \mathrm{d}\mu\,\mathrm{d}s$$

where we used the property of the semigroup and the definition of $f_{n,t}$. Hence, taking the expectation and using indipendence:

$$\begin{aligned}
\mathbb{E}\,\|f_{n,t} - 1\|^2_{H_\mu^{-1}(M)} &= 2\int_t^\infty \int_M \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}[p_s(X_i, y) - 1]\right]^2 \mathrm{d}\mu(y)\,\mathrm{d}s \\
&= \frac{2}{n}\int_0^\infty \int_M \mathbb{E}([p_s(X_i, y) - 1]^2)\,\mathrm{d}\mu(y)\,\mathrm{d}s \\
&= \frac{2}{n}\int_t^\infty \int_M \int_M [p_s(x, y) - 1]^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,\mathrm{d}s \\
&= \frac{1}{n}\int_{2t}^\infty \int_M [p_s(x, x) - 1]\,\mathrm{d}\mu(x)\,\mathrm{d}s
\end{aligned}$$

where in the last step we used Equation 1.8.

Summing up, we proved the following upper bound on the minimal cost.

Theorem 7. *For any fixed $t > 0$ and $n \in \mathbb{N}$, it holds:*

$$\mathbb{E}\left[W_2^2(\mu_n, \mu)\right] \lesssim \int_M \int_M p_t(x, y)|x - y|^2\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) + \frac{1}{n}\int_{2t}^\infty \int_M [p_s(x, x) - 1]\,\mathrm{d}\mu(x)\,\mathrm{d}s.$$

where " $\lesssim$ " underlines that the bound is made up to a positive constant.

We will denote by:

$$D_t^2 := \int_M \int_M p_t(x, y)|x - y|^2\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)$$

the so-called *dispersion factor* and with:

$$E_t^2 := \frac{1}{n}\int_{2t}^\infty \int_M [p_s(x, x) - 1]\,\mathrm{d}\mu(x)\,\mathrm{d}s$$

the other contribution.

The idea is then to use Theorem 7 and estimate the two energies D_t^2, E_t^2 , eventually optimizing in the choice of $t > 0$. As a consequence of the properties of the kernel p_t, what one usually has is a sublinear behaviour in t for the dispersion factor (using compactness assumptions on M for example, but also in the Gaussian case), namely:

$$D_t^2 \leq Ct, \quad 0 < t \leq 1, \quad C \in \mathbb{R}^+;$$

on the other hand, in the second energy E_t^2 the important part is the local behaviour in $t \to 0$ of

$$\int_M [p_s(x, x) - 1]\,\mathrm{d}\mu(x)\,\mathrm{d}s;$$

one more time, the study of such a behaviour will be central in the asymptotic scale of the minimal cost, similarly to how it is in the [2] work (in fact, the approach in [2] is very similar: here we weren't interested in the exact limit but only in the upper bound). The key property of (M, g, μ) in order to obtain the expected behaviours (as in the uniform case) is a Poincaré inequality, namely:

$$\int_M \varphi^2 \, \mathrm{d}\mu \leq C \int_M |\nabla \varphi|^2 \, \mathrm{d}\mu$$

for some $C = C(d) > 0$.

Theorem 8 (Ledoux, [6], Theorem 4)**.** *Assume that the dispersion factor D_t^2 is sublinear in small time, that is $D_t^2 \leq C_U t$ for $0 < t \leq 1$, and that μ satisfies a Poincaré inequality with constant $C_P > 0$. Suppose the kernel p_t satisfies the local bound:*

$$p_t(x, y) \leq C_u t^{-\frac{d}{2}}, \quad 0 < t \leq 1, \ x, y \in M; \tag{1.21}$$

then we have:

$$\mathbb{E}\left[W_2^2(\mu_n, \mu)\right] \leq \begin{cases} C\frac{log n}{n} & \text{if } d = 2, \\ Cn^{-\frac{2}{d}} & \text{if } d \geq 3, \end{cases} \tag{1.22}$$

when C depends only on C_D, C_P, C_u, d.

The proof of the previous theorem is mainly based on the previous computations we showed: the key part is to estimate the energy E_t^2 (as a consequence of the Poincaré inequality and the local bound of the kernel p_t) and then optimize in $t > 0$.

1.2.2 The Gaussian case: $d = 2$, $p = 2$.

The previous theorem covers a huge amount of different cases but it still leaves out the one of Gaussian samples. In particular, neither Theorem 8 nor previous works in literature are able to cover the apparently simple case of Gaussian samples in $\mathbb{R}^2$ with $p = 2$.
Precisely, let us consider $M = \mathbb{R}^d$ and $\mu = Ce^{-\frac{|x|^2}{2}}$ as the (renormalized) Gaussian measure in $\mathbb{R}^d$; note that Theorem 7 has been proved in a very general setting and applies in this case too. The semigroup generated by the Gaussian measure is called *Ornstein-Uhlenbeck* and the correspondent *Mehler* kernel can be explicitly computed and it is defined by:

$$\int_{\mathbb{R}^d} p_t(x, y)\varphi(y) \, \mathrm{d}\mu(y) = \int_{\mathbb{R}^d} \varphi\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) \, \mathrm{d}\mu(y) = P_t\varphi(x)$$

for any $t > 0$ and $x \in \mathbb{R}^d$. In particular, one has:

$$p_t(x, x) = \frac{1}{(1 - e^{-2t})^{\frac{d}{2}}} \exp\left(\frac{e^{-t}}{1 + e^{-t}}|x|^2\right), \quad t > 0, \ x \in \mathbb{R}^d. \tag{1.23}$$

Using these explicit formulas, it's not difficult to see that the dispersion factor D_t^2 is still sublinear in small time, even if we are not working on a compact space:

$$\begin{aligned} D_t^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 p_t(x, y) \, \mathrm{d}\mu(x \, \mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left|x - e^{-t}x + \sqrt{1 - e^{-2t}}y\right|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \leq 2dt \end{aligned}$$

for any $0 < t \leq 1$.

Unfortunately, on the other hand the estimate of the energy E_t^2 is less efficient: indeed, using that for any $s > 0$

$$\int_{\mathbb{R}^d} p_s(x,x)\,\mathrm{d}\mu(x) = \frac{1}{(1-e^{-s})^d}$$

we can only deduce:

$$\int_t^\infty \int_{\mathbb{R}^d} [p_s(x,x) - 1]\,\mathrm{d}\mu(x)\,\mathrm{d}t \leq \int_t^\infty \frac{de^{-s}}{(1-e^{-s})^d}\,\mathrm{d}s \leq \begin{cases} Clog\left(\frac{1}{t}\right) & \text{if} \quad d = 1, \\ Ct^{-d} & \text{if} \quad d \geq 2, \end{cases}$$

for some constant $C > 0$; after the optimization in $t > 0$, we end up with:

$$\mathbb{E}\left[W_2^2(\mu_n, \mu)\right] = \begin{cases} O\left(\frac{logn}{n}\right) & \text{if} \quad d = 1, \\ O\left(n^{-\frac{1}{d}}\right) & \text{if} \quad d \geq 2. \end{cases}$$

Unfortunately, these bounds are not of the expected order, for example according to the known result in dimension $d = 1$. Here the key missing property is a suitable local behaviour, for small time, of the Mehler kernel:

$$p_t(x,x) \sim e^{\frac{|x|^2}{2}} s^{-\frac{d}{2}}, \quad x \in \mathbb{R}^d,$$

when $s \to 0$. Unfortunately $x \to \exp\{\frac{|x|}{2}\}$ is not in $L^1_\mu(\mathbb{R}^d)$ and this represents an obstacle into the proof of Theorem 8.

The main idea proposed by Ledoux in [6] in order to improve the previous estimates and try to overcome the difficulties of a non compactly supported measures is to work with *localized Gaussian measures*. We just introduce the basic idea but we omit here all the computations. The setting is the following: let us fix an $R > 0$ and consider:

$$\mathrm{d}\mu^R := \frac{1}{\mu(B_R)} \mathbb{1}_{B_R}\,\mathrm{d}\mu$$

as our localized Gaussian measure on the ball of radius $R > 0$. Define a new sequence of random variables given by:

$$X_i^R := \begin{cases} X_i & \text{if} \quad |X_i| \leq R, \\ Z_i & \text{if} \quad |X_i| > R, \end{cases}$$

where $(Z_i)_i$ are independent random variables with law μ^R, independent to the X_i. In particular, X_i^R takes value only in the ball B_R; the idea is then try to apply the classical theory to this localized measure and compare the error between this and the original problem, as $R \to +\infty$, possibly in a suitable dependence on $n \in \mathbb{N}$ and $t > 0$. To do it, let us set:

$$\mu_n^R := \frac{1}{n} \sum_{i=1}^n \delta_{X_i^R}$$

for any $n \in \mathbb{N}$ and

$$\mathrm{d}\mu_{n,t}^R := \frac{1}{n} \sum_{i=1}^n p_t(X_i, \cdot)\,\mathrm{d}\mu$$

for $n \in \mathbb{N}$ and $t > 0$. First of all, one can observe that:

$$\mathbb{E}\left[W_2^2(\mu_n, \mu_n^R)\right] \leq 4 \int_{|x|>R} |x|^2\,\mathrm{d}\mu$$

which is of the order of $R^d e^{-\frac{R^2}{2}}$ as $R \to +\infty$. Consequently, for the natural choice of $R := c\sqrt{logn}$ for any $c > 0$, the previous estimate yields:

$$\mathbb{E}\left[W_2^2(\mu_n, \mu_n^R)\right] = O\left(\frac{1}{n^{c'}}\right) \tag{1.24}$$

where $2c' = c^2$. In particular, with this choice of R we can get any suitable polynomial bound, choosing a $c > 0$ large enough.

On the other side, with similar computations as in the non-localized case, one can prove:

$$\mathbb{E}\left[W_2^2(\mu_n^R, \mu_{n,t}^R)\right] \leq 4dt \tag{1.25}$$

for $R > 0$ sufficiently large, $t > 0$ and $n \in \mathbb{N}$.

Finally, the last part we have to estimate is the distance between the regularized and localized measures $\mu_{n,t}^R$ and μ. Again we make use of Proposition 3 to get:

$$\mathbb{E}\left[W_2^2(\mu_{n,t}^R, \mu)\right] \leq 4\mathbb{E}\left\|f_{n,t}^R - 1\right\|_{H_\mu^{-1}(\mathbb{R}^d)};$$

therefore, using properties of the semigroup and the Poincaré inequality for the Gaussian measure in $\mathbb{R}^d$, it is possible to prove:

$$\mathbb{E}\left[W_2^2(\mu_{n,t}^R, \mu)\right] \leq C\left(\frac{1}{2n}\int_{2t}^{\infty}\int_{\mathbb{R}^d}\left[p_s(x,x) - 1\right]\,\mathrm{d}\mu^R\,\mathrm{d}s\right) \tag{1.26}$$

where again $R \sim c\sqrt{logn}$ and $C > 0$. Putting together (1.24), (1.25) and (1.26), we obtain:

$$\mathbb{E}\left[W_2^2(\mu_n, \mu)\right] \leq C\left(\frac{1}{n^{c'}} + t + \frac{1}{2n}\int_{2t}^{\infty}\int_{\mathbb{R}^d}\left[p_s(x,x) - 1\right]\,\mathrm{d}\mu^R\,\mathrm{d}s\right) \tag{1.27}$$

for $R \sim c\sqrt{logn}$, $C > 0$, $c' > 1$ and $t > 0$. Note that such a bound is very similar, up to an small error, to the one proved in the unlocalized case but with μ^R instead of μ. Unfortunately, after a suitable optimization in $t > 0$, the previous estimate is only helpful to prove:

$$\mathbb{E}\left[W_2^2(\mu_n, \mu)\right] = O\left(\frac{(logn)^2}{n}\right)$$

as $n \to +\infty$. It is still an open problem to understand the correct behaviour in the Gaussian case in dimension 2; the analysis in the compact case may suggest the correct bound to be of the order:

$$\frac{logn}{n}$$

but the extra logarithmic factor found by Ledoux with the localized approach seems to be a difficult point to treat.

1.2.3 Open problems and possible direction

Within the setting of the random matching problems there are a huge number of variables such as the dimension of the space, the cost function and obviously the law of the samples. There are still a lot of cases, in particular related to the non compact cases, that are not completely understood or at least not approached rigorously. The physic community has had a very strong impact in this framework, with lots of works devoted to random assignment problems; a very important example of these contributions is the work of Caracciolo, Lucibello, Parisi and Sicuro [3], in which the limit behaviour proved in [2] was firstly introduced, on the basis of a formal computation based on the linearization of the Monge-Ampere equation. Among the results proved (at least numerically) in [3], there are also some interesting guesses concerning the high order approximation of the average limit as $n \to +\infty$.

Theorem 9 (CLPS)**.** *Let* $X = [0,1]^d$ *and* $p = 2$. *Let* μ_n *and* ν_n *be the two empirical measures (related to* X_i *and* Y_j*). Then it holds:*

$$n^{\frac{2}{d}}\mathbb{E}\left[W_2^2(\mu_n,\nu_n)\right] \approx \begin{cases} \frac{n}{3} & \text{if} \quad d=1 \\ \frac{1}{2\pi} logn & \text{if} \quad d=2 \\ C_{d,2} + \frac{\zeta_d(1)}{2\pi^2} n^{(2-d)/d} & \text{if} \quad d \geq 3 \end{cases}$$

where $\zeta_d(x)$ *is the Epstein zeta function and the constants are validated numerically.*

The proof of Ambrosio Stra and Trevisan strongly relies on the linearization idea of [3]; a very challenging possibility could be to try to prove rigorously the other guesses of [3], eventually using a similar PDE approach as the one used in [2].

Another possible direction is definitely to look deeper into the properties of:

$$W_2^2(\mu_n,\mu)$$

as a random variable, that means not to limit the study to a simple average analysis. For example, it would be very interesting to study the oscillation (or the variance) and eventually try to get some deeper result of a possible distributional limit as $n \to +\infty$. Nevertheless, we want to point out that it seems unreasonable to expect sharp upper bounds for the variance working in the same spirit of Ledoux work, even if we believe that such bounds can be proved anyway; on the other hand, a more precise analysis of the lower bounds for the variance has been proposed by Chatterjee in [4]. We will see how to apply such a theory to the random assignment problem in the last chapter of this work.

Finally, a natural but more complicated issue would be to analyse the distributional behaviour of the minimum; that is, is it possible to find:

$$\frac{n}{logn} W_2^2(\mu_n,\mu) \xrightarrow{\mathcal{L}} Y$$

for a suitable random variable Z? Obviously, a first analysis of the oscillations seems to be necessary to get a clue for that.

Chapter 2

Talagrand and Poincaré inequalities

In this chapter, we gives a possible proof of the Poincaré inequality under the assumption that a Talagrand inequality holds; the proof is based on the study of the infinitesimal behaviour of the Wasserstein distance which plays an important role in the study of the random matching as well (in some sense, Lemma 11 shows the sharpness of the dual bond seen in Proposition 3).

Let $\Omega \subset \mathbb{R}^n$ and $\mu \in \mathscr{P}_2(\Omega)$. The $\mu-$*Relative Entropy* is the functional defined by:

$$\mathbf{H}[\nu, \mu] = \begin{cases} \int_\Omega f \log f \,\mathrm{d}\mu & \text{if } \nu << \mu, \quad \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = f; \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Theorem 10. *Let's suppose μ to satisfy Talagrand inequality, namely:*

$$W_2^2(\mu, \nu) \leq 2C\mathbf{H}[\nu, \mu], \quad \forall \nu << \mu. \tag{2.2}$$

Then μ satisfies the Poincare' inequality with constant $C > 0$:

$$\int_\Omega g^2 \,\mathrm{d}\mu \leq C \int_\Omega |\nabla g|^2 \,\mathrm{d}\mu \tag{2.3}$$

for any $g \in H^1(\Omega) \cap ImL_\mu$ with $\int_\Omega g \,\mathrm{d}\mu = 0$, where L_μ is the operator generated by μ defined in Equation 1.5.

The general idea behind this result is to use a "Taylor-type" expansion next to μ and to estimate the local behaviour of the right-hand-side and left-hand-side of (2.2) (as suggested in [7]). Let's take a smooth, compactly supported and zero $\mu-$mean $g : \Omega \to \mathbb{R}$ and define:

$$\mu_\epsilon \in \mathscr{P}_2(\Omega), \quad \mathrm{d}\mu_\epsilon = (1 + \epsilon g) \,\mathrm{d}\mu. \tag{2.4}$$

Note that thanks to our hypothesis on the function g these are still probability measures. Let's start to evaluate the quantity:

$$\mathbf{H}[\mu_\epsilon, \mu] = \int_\Omega f_\epsilon \log f_\epsilon \,\mathrm{d}\mu$$

where $f_\epsilon = 1 + \epsilon g$ denotes the density of μ_ϵ with respect to μ. Note that $\log(1+x) \approx x - \frac{x^2}{2}$ as $x \to 0$; this means that $\mathbf{H}[\mu_\epsilon, \mu] \approx \epsilon^2$. In fact:

$$\frac{\mathbf{H}[\mu_\epsilon, \mu]}{\epsilon^2} = \frac{1}{\epsilon^2} \int_\Omega (1 + \epsilon g)(\epsilon g - \frac{\epsilon^2}{2} g^2) \,\mathrm{d}\mu + o(1) = \frac{1}{2} \int_\Omega g^2 \,\mathrm{d}\mu + o(1) \tag{2.5}$$

where we have already used that $\int_\Omega g \, \mathrm{d}\mu = 0$.

On the other hand, the key result we have to prove concerning the left-hand-side of (2.2) is the following.

Lemma 11. *Given μ and μ_ϵ as before, we have that:*

$$\lim_{\epsilon \to 0} \frac{W_2^2(\mu_\epsilon, \mu)}{\epsilon^2} = ||g||^2_{H^{-1,2}(\mu)} \tag{2.6}$$

where $||\cdot||_{H^{-1,2}(\mu)}$ denotes the dual norm in $H_0^1(\Omega, \mu)$ defined by:

$$||g||_{H^{-1,2}(\mu)} := \sup \left\{ \int_\Omega g\varphi \, \mathrm{d}\mu \ : \ |\varphi|_{H_0^1(\Omega,\mu)} \le 1 \right\} \tag{2.7}$$

Proof. The main tool we will use to prove the lemma is going to be the *Benamou-Brenier formula*:

$$W_2^2(\mu_1, \mu_0) = \inf \left\{ \int_0^1 \int_\Omega |v_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \ : \ \frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \operatorname{div}(\mu_t \cdot v_t) = 0, \quad v_t \in L^2(\mu_t, \mathbb{R}^n) \right\}. \tag{2.8}$$

The differential equation in the space of measures has to be tought in a distributional sense and it is called *continuity equation* (CE). In particular, we work with $\mu_1 = \mu_\epsilon$ and $\mu_0 = \mu$; note that the previous formula can be equivalently rewritten as:

$$W_2^2(\mu_\epsilon, \mu_0) = \inf \left\{ \epsilon \int_0^\epsilon \int_\Omega |v_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \ : \ \frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \operatorname{div}(\mu_t \cdot v_t) = 0, \quad v_t \in L^2(\mu_t, \mathbb{R}^n) \right\}. \tag{2.9}$$

Let's fix any $\varphi \in C_c^\infty(\Omega)$ with $\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}\mu \le 1$; actually, we will suppose something stronger than this [1], namely:

$$\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}\mu_t \le 1, \quad 0 \le t \le 1.$$

Then Holder inequality yields:

$$\int_\Omega |v_t|^2 \, \mathrm{d}\mu_t \ge \left(\int_\Omega \langle v_t, \nabla\varphi \rangle \, \mathrm{d}\mu_t \right)^2. \tag{2.10}$$

Integrating (2.10) and using Jensen inequality we get:

$$\int_0^1 \int_\Omega |v_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \ge \left(\int_0^1 \int_\Omega \langle v_t, \nabla\varphi \rangle \, \mathrm{d}\mu_t \, \mathrm{d}t \right)^2.$$

Suppose now $(\mu_t, v_t)_t$ solve (CE); hence from the latter inequality we obtain:

$$\int_0^1 \int_\Omega |v_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \ge \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varphi \, \mathrm{d}\mu_t \, \mathrm{d}t \right)^2 = \left(\int_\Omega \varphi \, \mathrm{d}\mu_\epsilon - \int_\Omega \varphi \, \mathrm{d}\mu_0 \right)^2$$

and using the definition of μ_ϵ we get:

$$\int_0^1 \int_\Omega |v_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \ge \epsilon^2 \int_\Omega \varphi g \, \mathrm{d}\mu \tag{2.11}$$

which gives us the sought lower bound as $\epsilon \to 0$.

[1] This is made by simplicity, using the formulation (2.9) it can be avoided.

On the other hand, to get the upper bound we have to find a family of solutions of (CE) which is controlled by the dual norm of the pertubation g in the limit as $\epsilon \to 0$. Fix $\epsilon > 0$ and consider the measure-valued curve:

$$\mathrm{d}\mu_t = (1 + tg)\,\mathrm{d}\mu, \quad t \in [0, \epsilon].$$

Let us take now a $\varphi \in C_c^\infty$ and compute:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varphi\,\mathrm{d}\mu_t = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varphi(1 + tg)\,\mathrm{d}\mu = \int_\Omega \varphi g\,\mathrm{d}\mu = \int_\Omega \langle \nabla G, \nabla \varphi \rangle\,\mathrm{d}\mu$$

where G is the unique-determined representation in the Hilbert space $H_0^1(\Omega)$ of the linear continuous functional:

$$\varphi \mapsto \int_\Omega \varphi g\,\mathrm{d}\mu. \tag{2.12}$$

This means, in particular:

$$||g||^2_{H^{-1,2}(\mu)} = \int_\Omega |\nabla G|^2\,\mathrm{d}\mu = |G|^2_{H_0^1(\Omega,\mu)} \tag{2.13}$$

Following what we obtained in (2.12), we deduce that defining:

$$\bar{v}_t := \frac{\nabla G}{1 + tg}, \quad t \in [0, \epsilon]$$

one ends up with a solution of (CE) given by the couple (μ_t, v_t). Note that $\bar{v}_t$ is well-defined, at least for small t. It follows from Benamou-Brenier formula:

$$W_2^2(\mu_\epsilon, \mu) \leq \epsilon \int_0^\epsilon \int_\Omega |\bar{v}_t|^2\,\mathrm{d}\mu_t\,\mathrm{d}t = \epsilon \int_0^\epsilon \int_\Omega \frac{|\nabla G|^2}{(1 + tg)^2}(1 + tg)\,\mathrm{d}\mu\,\mathrm{d}t. \tag{2.14}$$

Therefore, dividing by ϵ^2, we get:

$$\frac{W_2^2(\mu_\epsilon, \mu)}{\epsilon^2} \leq \int_\Omega |\nabla G|^2 \left(\fint_0^\epsilon \frac{1}{1 + tg}\,\mathrm{d}t \right) \mathrm{d}\mu$$

and then taking the limit in $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \frac{W_2^2(\mu_\epsilon, \mu)}{\epsilon^2} \leq \lim_{\epsilon \to 0} \int_\Omega |\nabla G|^2 \left(\fint_0^\epsilon \frac{1}{1 + tg}\,\mathrm{d}t \right) \mathrm{d}\mu = \int_\Omega |\nabla G|^2\,\mathrm{d}\mu = ||g||^2_{H^{-1,2}(\mu)}$$

where in the latter equality we used (2.13). Then using what we gained in (2.11), we obtain:

$$\lim_{\epsilon \to 0} \frac{W_2^2(\mu_\epsilon, \mu)}{\epsilon^2} = ||g||^2_{H^{-1,2}(\mu)}$$

which ends the proof. □

Finally, using Lemma 11 and Equation 2.5, Talagrand inequality (2.2) yields:

$$||g||^2_{H^{-1,2}(\mu)} \leq C \int_\Omega g^2\,\mathrm{d}\mu \tag{2.15}$$

for any $g \in C_c^\infty(\Omega)$ with $\int_\Omega g\,\mathrm{d}\mu = 0$. Let's recall the definition of the operator L:

$$\begin{gathered} L : \mathrm{D}(L) \subset L^2(\Omega, \mu) \to L^2(\Omega, \mu) \\ \int_\Omega \varphi(-L\psi)\,\mathrm{d}\mu = \int_\Omega \langle \nabla \varphi, \nabla \psi \rangle\,\mathrm{d}\mu, \quad \forall \varphi, \psi \in C_c^\infty(\Omega). \end{gathered} \tag{2.16}$$

Using the operator L, equation (2.15) becomes:

$$\int_\Omega |\nabla G|^2 \,\mathrm{d}\mu \leq C \int_\Omega |LG|^2 \,\mathrm{d}\mu, \quad \forall G \in \mathrm{D}(L). \tag{2.17}$$

Our goal is to prove:

$$\int_\Omega g^2 \,\mathrm{d}\mu \leq C \int_\Omega |\nabla g|^2 \,\mathrm{d}\mu,$$

where $g = LG$ with $LG \in H_0^1(\Omega)$. Under this assumption:

$$\int_\Omega g^2 \,\mathrm{d}\mu = \int_\Omega LG \cdot LG \,\mathrm{d}\mu = -\int_\Omega \langle \nabla LG, \nabla G \rangle \,\mathrm{d}\mu \leq \left(\int_\Omega |\nabla LG|^2 \,\mathrm{d}\mu \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla G|^2 \,\mathrm{d}\mu \right)^{\frac{1}{2}}.$$

Using that $g = LG$ and (2.17) we get:

$$\int_\Omega g^2 \,\mathrm{d}\mu \leq \left(\int_\Omega |\nabla g|^2 \,\mathrm{d}\mu \right)^{\frac{1}{2}} \left(C \int_\Omega |LG|^2 \,\mathrm{d}\mu \right)^{\frac{1}{2}} = \left(\int_\Omega |\nabla g|^2 \,\mathrm{d}\mu \right)^{\frac{1}{2}} \left(C \int_\Omega |g|^2 \,\mathrm{d}\mu \right)^{\frac{1}{2}}.$$

Dividing by $\left(\int_\Omega g^2 \,\mathrm{d}\mu\right)^{\frac{1}{2}}$ and taking the square we end up with:

$$\int_\Omega g^2 \,\mathrm{d}\mu \leq C \int_\Omega |\nabla g|^2 \,\mathrm{d}\mu$$

which ends the proof.

Chapter 3

Lower bound of fluctuations for random matching problem

In this last chapter, we introduce a very recent method proposed by Chatterjee in [4] to study lower bounds for fluctuations within a suitable general class of random problems (specifically homogenoous ones); firstly, let us introduce the following class of admissible measures.

Definition 12. Let's denote with $S(d)$ the space of all the functions $V \in C^{\infty}(\mathbb{R}^d)$ with every derivative decaying at most polynomially and such that e^V grows faster than any polynomial. Let define then:

$$P(d) := \left\{ \mu \in \mathscr{P}(\mathbb{R}^d) \ : \ \mu = e^{-V} \, \mathrm{d}x, \ V \in S(d) \right\}.$$

The key result proved by Chatterjee in [4] is the following theorem.

Theorem 13. *Let $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ be IID random variables with law in $P(d)$. Let $L_n : \mathbb{R}^{2n} \to \mathbb{R}^+$ be an $r-$homogeneous fucntion, with $r > 0$. Suppose to have:*

$$\liminf_{n \to +\infty} \mathbb{P}(L_n(X, Y) \geq t_n) > 0. \tag{3.1}$$

Then $L_n(X_1, ..., X_n, Y_1, ..., Y_n)$ has fluctations of order at least $t_n n^{-\frac{1}{2}}$.

We want to apply this results to get lower bound on the fluctuations for the RMP.

The first step is to prove the following general lemma which it will be useful in the proof of the main result of this section.

Lemma 14. *Let $(\Omega, \mathscr{P})$ be a probability space and take $A_1, ..., A_n \subset \Omega$ a sequence of events such that:*

$$\mathbb{P}(A_i) \geq \alpha, \quad \forall i = 1, ..., n \tag{3.2}$$

for some $\alpha \in \mathbb{R}^+$. Let's define:

$$S_n := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{A_i}, \quad n \geq 1.$$

Then for all $\beta \in (0, 1 \wedge \alpha)$ there exists another constant $\tilde{\alpha} = \tilde{\alpha}(\alpha, \beta) \in \mathbb{R}^+$ such that:

$$\mathbb{P}(S_n \geq \beta) \geq \tilde{\alpha}, \quad \forall n \geq 1. \tag{3.3}$$

Proof. Clearly, we have:

$$\mathbb{E}(S_n) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(A_i) \geq \alpha, \quad \forall n \geq 1. \tag{3.4}$$

Moreover, by definition we also get that $S_n \leq 1$, for all $n \geq 1$. Then it follows:

$$\begin{aligned} \mathbb{E}(S_n) &= \mathbb{E}(S_n \cdot \mathbb{1}_{S_n<\beta}) + \mathbb{E}(S_n \cdot \mathbb{1}_{S_n\geq\beta}) \\ &\leq \beta\mathbb{P}(S_n < \beta) + \mathbb{P}(S_n \geq \beta) \\ &= \beta + (1-\beta)\mathbb{P}(S_n \geq \beta). \end{aligned} \tag{3.5}$$

Finally, (3.4) and (3.5) yield:

$$\mathbb{P}(S_n \geq \beta) \geq \frac{\alpha-\beta}{1-\beta} =: \tilde{\alpha} \in \mathbb{R}^+, \quad \forall n \geq 1$$

which ends the proof. □

We are now ready to prove the key lemma to apply Chatterjee's Theorem 13.

Lemma 15. *Let $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ be IID random variables with law in $P(d)$. Let us consider the random matching problem given by:*

$$L_n(X_1, ..., X_n, Y_1, ..., Y_n) := \min_{\sigma\in\Sigma_n}\left\{\frac{1}{n}\sum_{i=1}^{n}|X_i - Y_{\sigma(i)}|\right\}. \tag{3.6}$$

Then we have:

$$\liminf_{n\to+\infty}\mathbb{P}(L_n(X,Y) \geq \delta n^{-\frac{1}{d}}) > 0$$

for a suitable $\delta \in \mathbb{R}^+$.

Proof. The first key observation is that a lower bound for the cost L_n is given by the sum of the closest neighborhood distances of the points, namely:

$$L_n(X_1, ..., X_n, Y_1, ..., Y_n) \geq \frac{1}{n}\sum_{i=1}^{n}d(X_i, \mathcal{Y}) \tag{3.7}$$

where $\mathcal{Y} = \{Y_1, ..., Y_n\}$ is the set of $Y-$points and:

$$d(X_i, \mathcal{Y}) = \min_{j=1...n}|X_i - Y_j|$$

denotes the minimal distance of X_i for the $Y-$points. To prove Equation 3.7, note that if the $n-$minima in the right-hand side sum of Equation 3.7 are all realized by different $Y_{j(i)} \in \mathcal{Y}$, then this assignment is an admissible one for the RMP and actually it is the best one (we get an equality in Equation 3.7); if there are at least two equal $Y_{j(i)}$, then this means that any other bijective assignment would give a bigger value than the right-hand side of Equation 3.7.

Once we have Equation 3.7, we can deduce the thesis of the lemma looking at the one-to-one interactions between the points X and Y. In particular, let fix $i, j \in \{1, ..., n\}$ and consider the correspondent X_i, Y_j; we would like to study the set where:

$$|X_i - Y_j| < t_n$$

for a suitable T_n we will choose afterwards. Recalling that X_i and Y_j are independent and with the same law, we have:

$$\mathbb{P}(|X_i - Y_j| < t_n) = \iint\limits_{|x-y|<t_n} e^{-V(x)} e^{-V(y)} \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\mathbb{R}^d} e^{-V(x)} \int\limits_{B_{t_n}(x)} e^{-V(y)} \, \mathrm{d}y \, \mathrm{d}x. \tag{3.8}$$

Moreover, using the fact that densities in $P(d)$ are bounded, we also get:

$$\int\limits_{B_{t_n}(x)} e^{-V(y)} \, \mathrm{d}y \leq \left\| e^{-V} \right\|_\infty \mathscr{L}^d(B_{t_n}(x)) = \left\| e^{-V} \right\|_\infty \omega_d t_n^d$$

where w_n is the volume measure of the unit ball in $d-$dimension. Using this last inequality in Equation 3.8, we obtain:

$$\mathbb{P}(|X_i - Y_j| < t_n) \leq C t_n^d \tag{3.9}$$

where $C := \omega_n \left\| e^{-V} \right\|_\infty > 0$.

Therefore, using the single interaction estimate in Equation 3.9, we get:

$$\begin{aligned}\mathbb{P}\left(d(X_i, \mathcal{Y}) \geq t_n\right) = \mathbb{P}\left(d(X_i, Y_j) \geq t_n, \ \forall j = 1, ..., n\right) = 1 - \mathbb{P}\left(\bigcup_{=1}^{n} d(X_i, Y_j) < t_n\right) \\ \geq 1 - \sum_{j=1}^{n} \mathbb{P}\left(d(X_i, Y_j) > t_n\right) \geq 1 - nCt_n^d.\end{aligned}$$

Now if we choose:

$$t_n := \left(\frac{1}{2Cn}\right)^{\frac{1}{d}} = D\left(\frac{1}{n}\right)^{\frac{1}{d}}$$

we end up with:

$$\mathbb{P}\left(d(X_i, \mathcal{Y}) \geq Dn^{-\frac{1}{d}}\right) \geq \frac{1}{2} > 0 \tag{3.10}$$

for all $i = 1, ..., n$ and $n \in \mathbb{N}$. Hence, applying Lemma 14 with $\alpha := \frac{1}{2}$ and:

$$A_i := \left\{d(X_i, \mathcal{Y}) \geq Dn^{-\frac{1}{d}}\right\}$$

we obtain:

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} d(X_i, \mathcal{Y}) \geq \beta D n^{-\frac{1}{d}}\right) \geq \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} A_i \geq \beta\right) \geq \tilde{\alpha} > 0.$$

Recalling that (3.7) holds, choosing $\delta := \beta D$ we end up with:

$$\mathbb{P}(L_n(X, Y) \geq \delta n^{-\frac{1}{d}}) \geq \tilde{\alpha} > 0$$

that clearly implies our thesis.

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