1. Entanglement entropy for fermions in a magnetic field

A new tool that is usually used to analyze the quantum states that arise in condensed matter systems is the notion of entanglement, that is the physical phenomenon that occurs when groups of particles are generated or interact in ways such that the quantum state of each particle cannot be described independently of the others.

Thanks to this notion it was possible to bring a quantum information perspective also for traditional problems and technique in the field as quantum phase transitions, numerical simulation methods and renormalization group.

A measure of the entanglement is given by the Von Neumann entropy S_A of the reduced density matrix ρ_A of a part A of the total system, i.e. $S_A = -tr(\rho_A) \log \rho_A$. This quantity measures the amount of quantum entanglement of the subsystem A with its environment B.

If the quantum state has a finite correlation length then we can expect that the Von Neumann entropy is proportional to the area of the common boundary surface between A and B ("area law"), instead of all the volume of A.

Two dimensional systems with topological order (microscopically it corresponds to the patterns of long-range quantum entanglement) are particularly interesting for this kind of problems. In these cases we have that the area law for the entropy become $S_A = cP - \gamma + \mathcal{O}\left(\frac{1}{P}\right)$, where P is the perimeter of A, c is a non universal constant and γ is called topological entanglement entropy, that depends on the topological properties of the quantum state.

In this project our aim is to compute the entanglement entropy $S_{\mathcal{D}}$ for a disk $\mathcal{D}$ of radius R on a plane. In particular we consider pure states in which the ground state is given by free fermions in a magnetic field in $2D$.

We will show that for this model $\gamma = 0$ as it is predict by general arguments.

For the disk geometry, supposing that the magnetic length ℓ is equal to 1, the eigenfunctions for the lowest Landau level are:

$$\phi_m(z) = \frac{z^m}{(\pi m!)^{\frac{1}{2}}} e^{-\frac{|z|^2}{2}}, \qquad m \geq 0. \tag{1}$$

Since we have that

$$\chi_{\mathcal{D}} P \chi_{\mathcal{D}} = \sum_{m\geq 0} |\chi_{\mathcal{D}}\phi_m\rangle\langle\chi_{\mathcal{D}}\phi_m| = \sum_{m\geq 0} \lambda_m \frac{|\chi_{\mathcal{D}}\phi_m\rangle\langle\chi_{\mathcal{D}}\phi_m|}{\|\chi_{\mathcal{D}}\phi_m\|^2},$$

where $P = \sum_{m\geq 0} |\phi_m\rangle\langle\phi_m|$ and $\chi_{\mathcal{D}}$ is the projection on the disk $\mathcal{D}$, we can obtain the eigenvalues of the two points correlation matrix in the following way:

$$\lambda_m = \int_{\mathcal{D}} |\phi_m(z)|^2 \, dz, \quad m \geq 0,$$

that for this particular case assume the form

$$\lambda_m(R) = \frac{1}{m!}\left(\Gamma(m+1) - \Gamma\left(m+1, \frac{R^2}{2}\right)\right), \quad m \geq 0 \tag{2}$$

where $\Gamma(s,x)$ is the upper incomplete Gamma function.

Hence we can write the entanglement entropy $S_{\mathcal{D}}$ in the following way:

$$S_{\mathcal{D}} = \sum_{m\geq 0} H(\lambda_m),$$

where $H(x) := -x\log(x) - (1-x)\log(1-x)$.

More precisely we want to prove that the disk entanglement entropy follows a "perimeter law":

$$S_{\mathcal{D}} = \tilde{c}R \;\; as \; R \to +\infty,$$

with $\tilde{c} > 0$ a constant we will determine later.

Furthermore we will prove the following stronger Theorem:

Theorem 1.1. *Let $H : [0,1] \to \mathbb{R}$ defined as above then there exist $\tilde{c}, C > 0$ constants such that*

$$\left|\frac{1}{R}\sum_{m\geq 0} H(\lambda_m) - \tilde{c}\right| \leq \frac{C}{R^2}.$$

Remark 1.2. *Actually it is possible to prove a stronger result than the previous theorem, but we stated Theorem 1.1 in this way since we will prove Lemma 1.6, that is the lemma we will use to have the final estimate in Theorem1.1., only for $k = 2$. So it is possible to prove that*

$$\left|\frac{1}{R}\sum_{m\geq 0} H(\lambda_m) - \tilde{c}\right| \leq \frac{C}{R^k},$$

for each $k \geq 1$ if someone prove rigorously Lemma 1.6 for each $k \geq 1$.

The idea to prove Theorem 1.1 is to apply to Von Neumann entropy the Poisson summation formula:

$$\sum_{p\in\mathbb{Z}} f(p) = \sum_{\xi\in\mathbb{Z}} \hat{f}(\xi). \tag{3}$$

As a first step for the proof of Theorem 1.1 we start writing the eigenvalues in a different way.

Thanks to the definition of Gamma function on the integers we have that $\lambda_m(R) = 1 - \frac{1}{m!}\Gamma\left(m+1, \frac{R^2}{2}\right)$ and then after some computations:

$$\lambda_m(R) = 1 - e^{\frac{-R^2}{2}} \sum_{k=0}^{m} \left(\frac{R^2}{2}\right)^k \cdot \frac{1}{k!} \tag{4}$$

where this equality follows from the definition of upper incomplete Gamma function.

Now we want to understand the behavior of $\lambda_m(R)$ for R large enough and m of the form $m = \frac{R^2}{2} + Rx$, with $|x| \leq c\log R$ and $x \in \frac{\mathbb{Z} - \left(\frac{R^2}{2} - \left[\frac{R^2}{2}\right]\right)}{R}$, where $c > 0$ is a constant and $\forall y \in \mathbb{R}$ with $[y]$ we mean $[y] := p$, where p is the smallest integer such that $y \in [p, p+1)$.

Without loss of generality in the following computations we assume that $\frac{R^2}{2}$ is an integer and so $x \in \frac{\mathbb{Z}}{R}$ to simplify the notation; otherwise we consider $x \in \frac{\mathbb{Z} - \left(\frac{R^2}{2} - \left[\frac{R^2}{2}\right]\right)}{R}$ as we described above.

This means we are studying the behavior of the eigenvalues λ_m such that $m \in I := [\frac{R^2}{2} - cR\log R, \frac{R^2}{2} + cR\log R] \cap \mathbb{Z}$.

To do this we start with a simple estimate for eigenvalues λ_m such that $m \leq \frac{R^2}{2} - cR\log R$ or $m \geq \frac{R^2}{2} + cR\log R$:

Lemma 1.3. *If $m = \frac{R^2}{2} + Rx$ with $|x| \geq c\log R$ then*

$$\begin{cases} \lambda_m(R) = 1 - \mathcal{O}\left(e^{-2x^2}\right) & \text{if } x \leq c\log R \\ \lambda_m(R) = \mathcal{O}\left(e^{-2x^2}\right) & \text{if } x \geq c\log R \end{cases} . \tag{5}$$

Proof. We consider the case $x < 0$.

For this value of m we have:

$$\begin{aligned} \lambda_m(R) &= 1 - e^{-\frac{R^2}{2}} \sum_{k=0}^{\frac{R^2}{2}+Rx} \left(\frac{R^2}{2}\right)^k \cdot \frac{1}{k!} \\ &\geq 1 - Re^{-\frac{R^2}{2}} \left(\frac{R^2}{2}\right)^{\frac{R^2}{2}+Rx} \cdot \frac{1}{(\frac{R^2}{2}+Rx)!} \cdot \\ &\sim 1 - \cdot \frac{e^{-Rx}}{\left(1+\frac{2x}{R}\right)^{\frac{R^2}{2}+Rx}} \\ &\sim 1 - e^{-2x^2} \end{aligned}$$

In the first inequality we have used that $\sum_{k=0}^{\frac{R^2}{2}+Rx-dR}\left(\frac{R^2}{2}\right)^k\cdot\frac{1}{k!}$ goes to zero faster than e^{-2x^2} for R large enough and then in the last estimate we have used the Stirling formula for $\left(\frac{R^2}{2}-Rx\right)!$.

If $x>0$ the statement follows with a similar proof. $\square$

Proposition 1.4. *For $x\in\frac{\mathbb{Z}}{R}$ such that $|x|\leq c\log R$ and $m=\frac{R^2}{2}+Rx$ we have that:*

$$\lambda_m(R)=1-\sqrt{\frac{2}{\pi}}\sum_{k=-\infty,\,k\in\frac{\mathbb{Z}}{R}}^{x}\frac{e^{-2\left(\frac{k}{R}\right)^2}}{R}+\mathcal{O}\left(\frac{1}{R^{2c^2}}\right). \tag{6}$$

Proof. For the estimates in Lemma 1.3 we have that:

$$\begin{aligned}
\lambda_m(R)&=1-e^{\frac{-R^2}{2}}\sum_{k=0}^{m}\left(\frac{R^2}{2}\right)^k\cdot\frac{1}{k!}\\
&=1-e^{\frac{-R^2}{2}}\sum_{k=\frac{R^2}{2}-cR\log R}^{\frac{R^2}{2}+Rx}\left(\frac{R^2}{2}\right)^k\cdot\frac{1}{k!}+\mathcal{O}\left(\frac{1}{R^{2c^2}}\right)\\
&=1-e^{\frac{-R^2}{2}}\sum_{k=-c\log R,\,k\in\frac{\mathbb{Z}}{R}}^{x}\left(\frac{R^2}{2}\right)^{\frac{R^2}{2}+Rk}\cdot\frac{1}{\left(\frac{R^2}{2}+Rk\right)!}+\mathcal{O}\left(\frac{1}{R^{2c^2}}\right)\\
&\sim 1-e^{\frac{-R^2}{2}}\sum_{k=-c\log R,\,k\in\frac{\mathbb{Z}}{R}}^{x}\left(\frac{R^2}{2}\right)^{\frac{R^2}{2}+Rk}\cdot\\
&\cdot\frac{e^{\frac{R^2}{2}+Rk}}{\left(\frac{R^2}{2}+Rk\right)^{\frac{R^2}{2}+Rk}\sqrt{2\pi\left(\frac{R^2}{2}+Rk\right)}}+\mathcal{O}\left(\frac{1}{R^{2c^2}}\right)\\
&\sim 1-\sum_{k=-c\log R,\,k\in\frac{\mathbb{Z}}{R}}^{x}\frac{1}{e^{2k^2}\sqrt{2\pi\left(\frac{R^2}{2}+Rk\right)}}+\mathcal{O}\left(\frac{1}{R^{2c^2}}\right)
\end{aligned}$$

$$= 1 - \sum_{k=-c\log R,\, k\in\frac{\mathbb{Z}}{R}}^{x} \frac{e^{-2k^2}}{R\sqrt{\pi}\cdot\sqrt{1+\frac{2k}{R}}} + \mathcal{O}\left(\frac{1}{R^{2c^2}}\right)$$
$$\sim 1 - \sqrt{\frac{2}{\pi}} \sum_{k=-c\log R,\, k\in\frac{\mathbb{Z}}{R}}^{x} \frac{e^{-2k^2}}{R} + \mathcal{O}\left(\frac{1}{R^{2c^2}}\right) \qquad .$$
$$\sim 1 - \sqrt{\frac{2}{\pi}} \sum_{k=-\infty,\, k\in\frac{\mathbb{Z}}{R}}^{x} \frac{e^{-2k^2}}{R} + \mathcal{O}\left(\frac{1}{R^{2c^2}}\right)$$

We obtain the statment of this Lemma with the change of variable $h = Rk$, and hence $h \in \mathbb{Z}$. □

To understand the behavior of $\lambda_m(R)$ for $m \in I$ and R large enough we want to find an explicit formula for

$$\lambda_\infty(x) := \lim_{R\to\infty} \lambda_{\frac{R^2}{2}+Rx}(R),$$

that is

$$\lambda_\infty(x) := 1 - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{x} e^{-2y^2}\, dy. \tag{7}$$

We will prove this statement in the following Lemma.

Lemma 1.5. *For each $|x| \le c\log R$ and $n \in \mathbb{N}$ and $m = \frac{R^2}{2} + Rx$ with we have that*

$$\lambda_m(R) = \lambda_\infty(x) + \mathcal{O}\left(\frac{2^n x^{n+1} e^{-2x^2}}{R^n}\right) \tag{8}$$

Proof. (Sketch) Here we write only a sketch of the proof of this Lemma, since we will use the same technique at the end of this essay to prove the statement in Theorem 1.1.

We prove this Lemma for $x < 0$, since if instead $x > 0$ we use the same technique to estimate $1 - \sum_{k=-\infty}^{[x]} \frac{e^{-2\left(\frac{k}{R}\right)^2}}{R}$ instead of $\sum_{k=-\infty}^{[x]} \frac{e^{-2\left(\frac{k}{R}\right)^2}}{R}$.

From (6) we have that $\lambda_m(R) = 1 - \sqrt{\frac{2}{\pi}} \sum_{k=-\infty}^{x} \frac{e^{-2\left(\frac{k}{R}\right)^2}}{R} + \mathcal{O}\left(\frac{1}{R^{2c^2}}\right)$ if $|x| \le c\log R$, then the result follows using the Poisson summation formula for the sum $\sum_{k\in\mathbb{Z}} \frac{e^{-2\left(\frac{k}{R}\right)^2}}{R}$, indeed, defining

$$h_R(k) := \frac{e^{-2\left(\frac{k}{R}\right)^2}}{R}$$

and $v_R(k) := h_R(k) \cdot \chi_R(k)$, where $\chi_R(k)$ is a proper smooth cutoff function of the interval $(-\infty, x)$ defined as

$$\chi_R(k) := \begin{cases} 1 \text{ if } k \leq [x] \\ 0 \text{ if } k \geq [x]+1 \end{cases} .$$

Since

$$\sum_{k=[x]}^{[x]+1} h_R(k)\chi_R(k) \leq 2\frac{e^{-2\left(\frac{[x]}{R}\right)^2}}{R},$$

we have that

$$\sum_{k=-\infty}^{[x]} h_R(k) = \sum_{k\in\mathbb{Z}} v_R(k) + \mathcal{O}\left(\frac{e^{-2\left(\frac{[x]}{R}\right)^2}}{R}\right)$$
$$= \hat{v}_R(0) + \sum_{\xi\in\mathbb{Z}\setminus\{0\}} \hat{v}_R(\xi R) + \mathcal{O}\left(\frac{e^{-2\left(\frac{[x]}{R}\right)^2}}{R}\right).$$

Furthermore thanks to Fourier transform theory we have that

$$\hat{v}_R(\xi) = \frac{\widehat{v_R^{(n)}}}{(2\pi i\xi)^n},$$

and so

$$|\hat{v}_R(\xi)| \leq \frac{2^n C}{|\xi|^n}\left|\int_{-\infty}^{x} y^n e^{-2y^2}\, dy\right| \sim \frac{2^n x^{n+1}e^{-2x^2}}{|\xi|^n}.$$

and so that for each $k \geq 0$

$$\sum_{k=-\infty}^{[x]} \frac{e^{-2\left(\frac{k}{R}\right)^2}}{R} = \int_{-\infty}^{x} e^{-2y^2}\, dy + \mathcal{O}\left(\frac{2^k x^{k+1}e^{-2x^2}}{R^k}\right).$$

In this way we obtain the proof of of this Lemma since for $|x| \leq C\log R$ we have that $\frac{C}{R^{2c^2}} \leq \frac{C2^k x^{k+1}e^{-2x^2}}{R^k}$ indeed we get a bound for the error of the difference $|\lambda_m(R) - \lambda_\infty(x)|$. □

As a conseguenge of this lemma and (6) follows that $\lambda_m = 1 - e^{-2x^2} + \mathcal{O}\left(\frac{1}{R^{2c^2}}\right)$ if $x < 0$ and that $\lambda_m \sim e^{-2x^2} + \mathcal{O}\left(\frac{1}{R^{2c^2}}\right)$ if $x > 0$, with $m = \frac{R^2}{2} + Rx$ and $|x| \leq c\log R$.

If a reader check the proof of the previous Lemma, it shows that we have this limit result only for x that is an integer over R, instead we want such kind of result for each real x such that $|x| \leq c\log R$. To do this we have to define $\lambda(x, R)$ for each $x \geq -\frac{R}{2}$.

Hence for these values of x we define $\lambda(x,R)$ as

$$\lambda(x,R) := 1 - \frac{\Gamma\left(\frac{R^2}{2} + Rx + 1, \frac{R^2}{2}\right)}{\Gamma\left(\frac{R^2}{2} + Rx + 1\right)}.$$

This is a well posed definition since Gamma and incomplete Gamma function are well defined on all the real positive axis.

From this definition follows that (8) is true for each $|x| \le c \log R$, indeed we have this limit result for each $x \in \mathbb{Q}$ and $\lambda(x,R)$ is decreasing as a function of x.

From now on we will write $\lambda(x)$ instead of $\lambda(x,R)$, remembering the dependence by R.

Since we want to apply the Poisson summation formula for the series that represents the entanglement entropy of the system we want an estimate for $(H \circ \lambda)^{(k)}$ and so to get a bound for the $k-th$ derivative of $\lambda(x)$.

For $k \ge 2$ we have that $H^{(k)}(x) := \frac{d^k H}{dx^k} = \frac{(-1)^{k-1}\cdot(k-2)!}{x^{k-1}} - \frac{(k-2)!}{(1-x)^{k-1}}$ and $H'(x) = -\log x + \log(1-x)$.

Lemma 1.6. *For all $k \ge 1$ there exists a constant $c_* > 0$ such that for each $|x| \le c \log R$ we have*

$$|\lambda^{(k)}(x)| \le c_* |x|^k e^{-2x^2}.$$

Proof. We will prove this Lemma only for $k = 1, 2$, then the statement will follow iterating these computations.

In the proof of this Lemma we will use $\tilde{\Gamma}$ to indicate the upper incomplete Gamma function.

In the following with γ', $\tilde{\Gamma}'$ and Γ' we mean respectively

$$\gamma'\left(\frac{R^2}{2} + Rx + 1, \frac{R^2}{2}\right),$$

$$\tilde{\Gamma}'\left(\frac{R^2}{2} + Rx + 1, \frac{R^2}{2}\right)$$

and

$$\Gamma\frac{R^2}{2} + Rx + 1.$$

For the definition of $\lambda(x)$ we have that:

$$\lambda' = R \cdot \frac{\gamma'\Gamma - \Gamma'\gamma}{\Gamma^2}.$$

In particular we can divide the problem to estimate this quantity into two parts: one for $x > 0$ and one for $x < 0$.

For $x > 0$ we have that

$$|\lambda'| = R \cdot \left| \frac{\Gamma'}{\Gamma} - \frac{\gamma'}{\gamma} \right| \cdot |\lambda|.$$

Furthermore from the general theory for Gamma function we have that

$$\frac{\Gamma'\left(\frac{R^2}{2} + Rx + 1\right)}{\Gamma\left(\frac{R^2}{2} + Rx + 1\right)} \sim \log\left(\frac{R^2}{2} + Rx\right),$$

then thanks to a simple computation we can show that $\gamma' \sim \log \frac{R^2}{2} \gamma$. Indeed

$$\begin{aligned}
\log \frac{R^2}{2} \geq \frac{\gamma}{\gamma'} \geq \frac{1}{2} \log \frac{R^2}{2} \cdot \frac{\int_{\frac{R}{\sqrt{2}}}^{\frac{R^2}{2}} t^{\frac{R^2}{2}+Rx} e^{-t}\, dt}{\int_0^{\frac{R^2}{2}} t^{\frac{R^2}{2}+Rx} e^{-t}\, dt} \\
= \frac{1}{2} \log \frac{R^2}{2} \left(1 - \frac{\int_0^{\frac{R}{\sqrt{2}}} t^{\frac{R^2}{2}+Rx} e^{-t}\, dt}{\int_0^{\frac{R^2}{2}} t^{\frac{R^2}{2}+Rx} e^{-t}\, dt}\right) \\
\sim \frac{1}{2} \log \frac{R^2}{2}.
\end{aligned}$$

Moreover if $x < 0$ then we have that

$$|\lambda'| = R \cdot \left| \frac{\Gamma'}{\Gamma} - \frac{\tilde{\Gamma}'}{\tilde{\Gamma}} \right| \cdot |1 - \lambda|,$$

and that

$$\frac{\tilde{\Gamma}'}{\tilde{\Gamma}} = \frac{\Gamma' - \gamma'}{\Gamma - \gamma} \sim \frac{\Gamma'}{(1 - e^{-2x^2})\Gamma} - \frac{\gamma'}{(e^{2x^2} - 1)\gamma} \sim \log \frac{R^2}{2}.$$

In this estimate we used that for $|x| \leq c \log R$ we have $\lambda_m \sim 1 - e^{-2x^2}$ if $x < 0$ and $\lambda_m \sim e^{-2x^2}$ if $x > 0$.

Hence we can conclude that for each $|x| \leq c \log R$ we have the following estimate

$$|\lambda'(x)| \leq R \cdot \left| \log \frac{R^2}{2} - \log\left(\frac{R^2}{2} + Rx\right) \right| \cdot e^{-2x^2} \sim c_* |x| e^{-2x^2}.$$

Computing the second derivative of λ we have

$$\begin{aligned}
\lambda'' &= R^2 \left(\frac{\gamma''\Gamma - \gamma\Gamma''}{\Gamma^2} - \frac{2\Gamma'\left(\gamma'\Gamma - \gamma\Gamma'\right)}{\Gamma^3} \right) \\
&= R^2 \left(\frac{\gamma''\Gamma - \gamma\Gamma''}{\Gamma^2} - \frac{2\Gamma'}{\Gamma} \cdot \lambda' \right),
\end{aligned} \tag{9}$$

and then using again the estimates for λ' we obtain that

$$|\lambda''(x)| \leq c_*|x|^2 e^{-2x^2}.$$

From this computations we can see that for each derivative of $\lambda(x)$ we obtain a $|x|$ more. In this way, iterating these computations, we conclude the proof of this Lemma. □

In this way we have shown that in I the derivative $(H \circ \lambda)^{(k)}$ is always bounded.

Indeed for each $k \geq 2$ we have that

$$\left|(H \circ \lambda)^{(k)}\right| \leq c_*(k-2)! \left(\frac{1}{\lambda(x)} + \frac{1}{1-\lambda(x)} \right) |x|^k e^{-2x^2}. \tag{10}$$

Let $g_R(x) := H\left(\lambda\left(\frac{R^2}{2} + Rx\right)\right)$, for $|x| \leq c \log R$ and 0 otherwise.

$g_R(x)$ is not a smooth function, so, since we want a fast decay for the Fourier Transform of the function we are using in the Poisson summation formula, we define the function $F_R := g_R \cdot \chi_R$ on all $\mathbb{R}$, where $\chi_R(x) := \tilde{\chi}\left(\frac{x - \frac{R^2}{2}}{R \log R}\right)$, with $\tilde{\chi}$ a $C^\infty(\mathbb{R})$ cutoff function defined in the following way

$$\tilde{\chi}(t) := \begin{cases} 1 \text{ if } |t| \leq 1 \\ 0 \text{ if } |t| \geq 2 \end{cases} .$$

In this way we have that

$$\begin{aligned} \frac{1}{R} \sum_{m \geq 0} H(\lambda_m) &= \frac{1}{R} \sum_{m = \frac{R^2}{2} - cR\log R}^{\frac{R^2}{2} + cR\log R} H(\lambda_m) + \mathcal{O}\left(\frac{1}{R^{2c^2-2}}\right) \\ &= \frac{1}{R} \sum_{m=-cR\log R}^{cR\log R} g_R\left(\frac{m}{R}\right) + \mathcal{O}\left(\frac{1}{R^{2c^2-2}}\right) \\ &= \frac{1}{R} \sum_{m=-cR\log R}^{cR\log R} F_R\left(\frac{m}{R}\right) + \mathcal{O}\left(\frac{1}{R^{2c^2-2}}\right). \end{aligned}$$

We get he error $\mathcal{O}\left(\frac{1}{R^{2c^2-2}}\right)$ in the previous equalities considering only $m \in I$ and not in all $\mathbb{Z}$, since

$$C \sum_{k=-\frac{R}{2},\, k \in \frac{\mathbb{Z}}{R}}^{c \log R} e^{-2k^2} \leq CR^2 \frac{1}{R^{2c^2}}.$$

Since we want an estimate for the Poisson summation formula we want to use that (since in Lemma 1.6 we proved the decay only for

$k = 2$) for each $\xi \in \mathbb{R}$ we have that $|\hat{F}_R(\xi)| \leq \frac{\|(F_R)''\|_1}{\xi^2}$. To have this kind of estimate we want to have a bound of the L^1 norm of the second derivative of F_R, that is $F_R'' = \chi_R'' \cdot g_R + 2\chi_R' \cdot g_R' + \chi_R \cdot g_R''$.

Hence we have that

$$\sum_{\xi \in \mathbb{Z} \setminus 0} \hat{F}_R(R\xi) \leq \frac{C\|(F_R)''\|_1}{R^2} \leq \frac{C}{R^2},$$

indeed each term in F_R'' is bounded in L^1 thanks to the estimate in (10).

Then thanks to Poisson summation formula we obtain

$$\begin{aligned}
\frac{1}{R}\sum_{m \in \mathbb{Z}} F_R\left(\frac{m}{R}\right) &= \hat{F}_R(0) + \sum_{\xi \in \mathbb{Z} \setminus 0} \hat{F}_R(R\xi) \\
&= \int_{\mathbb{R}} F_R(x)\,dx + \mathcal{O}\left(\frac{1}{R^2}\right) \\
&= \int_{\mathbb{R}} g_R(x) \cdot \chi(x)\,dx + \mathcal{O}\left(\frac{1}{R^2}\right) \\
&= \int_{-c\log R}^{c\log R} H(\lambda(x,R))\,dx + \mathcal{O}\left(\frac{1}{R^2}\right) \\
&= \int_{-\infty}^{+\infty} H(\lambda_\infty(x))\,dx + \mathcal{O}\left(\frac{1}{R^2}\right);
\end{aligned}$$

in the last equality we have used the estimate for $\lambda(x,R)$ in (8), where $\lambda_\infty(x)$ is defined as the limit of $\lambda(x,R)$ for R goes to infinity and has an explicit formula in (7). Furthermore we have used that the error we do taking the limit for $R \to +\infty$ in the last equality is smaller than $\frac{1}{R^2}$, indeed for each $n \in \mathbb{N}$, for Lemma 1.5, the error we do is:

$$2^{n+1}\int_{c\log R}^{+\infty} \frac{x^{n+1}e^{-2x^2}}{R^n}\,dx \leq C\frac{(\log R)^{n+1}e^{-2c^2(\log R)^2}}{R^n} \leq \frac{C}{R^2}.$$

Then we define $\tilde{c} := \int_{-\infty}^{+\infty} H(\lambda_\infty(x))\,dx$.

Thanks to these last equalities we conclude the proof of Theorem 1.1, indeed we have that

$$\begin{aligned}
\frac{1}{R}\sum_{m\geq 0} H(\lambda_m) &= \frac{1}{R}\sum_{m=-cR\log R}^{cR\log R} F_R\left(\frac{m}{R}\right) + \mathcal{O}\left(\frac{1}{R^{2c^2-2}}\right) \\
&= \frac{1}{R}\sum_{m\in\mathbb{Z}} F_R\left(\frac{m}{R}\right) + \mathcal{O}\left(\frac{1}{R^{2c^2-2}}\right) \\
&= \int_{-\infty}^{+\infty} H(\lambda_\infty(x))\,dx + \mathcal{O}\left(\frac{1}{R^2}\right) + \mathcal{O}\left(\frac{1}{R^{2c^2-2}}\right) \\
&= \tilde{c} + \mathcal{O}\left(\frac{1}{R^2}\right).
\end{aligned}$$