The Vector Dyson equation for a class of skew-triangular block matrices

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Abstract

We study a class of skew-triangular block random matrices and compute their self-consistent density of states. We show that the self-consistent density of states diverges as a power law with exponent $-\frac{n-1}{n+1}$ at 0, where $n \times n$ is the number of blocks in the matrix.

1 Introduction

For a Hermitian random matrix $H = (h_{ij})_{1\le i,j\le N}$ with independent entries (up to the symmetry) with mean values 0 and variances $S = (s_{ij})_{1\le i,j\le N}$ we have the vector Dyson equation

$$\frac{-1}{\mathbf{m}} = z + S\mathbf{m}$$

with $z, m_1, \ldots, m_N \in \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Here we abuse notation and write $\frac{-1}{\mathbf{m}} = (-1/m_1, \ldots, -1/m_N)$. This equation has a unique solution [1, Theorem 6.1.4]. Note that this equation has the symmetry

$$z \to -\overline{z} \quad \text{and} \quad \mathbf{m} \to -\overline{\mathbf{m}}.$$

Solving this equation we may find the self-consistent density of states

$$\rho(E) = \lim_{\eta\searrow 0} \frac{1}{\pi} \langle \operatorname{Im} \mathbf{m}(E + i\eta)\rangle = \lim_{\eta\searrow 0} \frac{1}{\pi N} \sum_{k=1}^{N} \operatorname{Im} m_k(E + i\eta).$$

This (deterministic) object will in general approximate the (random) eigenvalue distribution of H if N is large, see [1]. By the symmetry of the equation ρ is an even function. We will here study the vector Dyson equation for a class of $nN \times nN$ matrices with

$$S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1(n-1)} & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2(n-1)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{(n-1)1} & s_{(n-1)2} & \cdots & 0 & 0 \\ s_{n1} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

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where the s_{ij} are $N \times N$ matrices such that $\left(s_{ij}\right)_{kl} \neq 0$ if $i + j \leq n + 1$ and $s_{ij} = 0$ if $i + j > n + 1$. The vector Dyson equation then takes the form

$$\frac{-1}{\mathbf{m}_k} = z + \sum_{j=1}^{n+1-k} s_{kj}\mathbf{m}_j, \qquad k = 1, \ldots, n,$$

where we split $\mathbf{m}$ according to the blocks, i.e. $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$. We expect that for reasonable assumptions on the s_{ij} we have

Conjecture 1.1. *As $E \longrightarrow 0$ we have $\rho(E) \sim |E|^{-\frac{n-1}{n+1}}$.*

Here we write $f(x) \sim g(x)$ as $x \longrightarrow 0$ if there exists constants $c, C > 0$ with $c|g(x)| \leq |f(x)| \leq C|g(x)|$ for all sufficiently small x.

2 Constant variance

We will first consider the case where $\left(s_{ij}\right)_{kl}$ are all the same value, which by scaling we may take to be $\left(s_{ij}\right)_{kl} = 1$. In this case all components of $\mathbf{m}_k$ are the same - they satisfy the same equation. We will thus just write this component as m_k. Similarly the action of s_{ij} is just multiplication of the value of any of its (all identical) entries 1. Thus, we will just treat it as a number $s_{ij} = 1$ if $i+j \leq n+1$ and $s_{ij} = 0$ otherwise. The vector Dyson equation thus becomes

$$\frac{-1}{m_k} = z + m_1 + \ldots + m_{n+1-k}, \qquad k = 1, \ldots, n. \tag{2.1}$$

Then $\rho(E) = \text{const.} \lim_{\eta \searrow 0} \sum_{k=1}^{n} \operatorname{Im} m_k(E + i\eta)$ In this setting we prove the conjectured result.

Theorem 2.1. *In the limit $E \longrightarrow 0$ we have that $\rho(E) \sim |E|^{-\frac{n-1}{n+1}}$.*

The main technical result we show is

Proposition 2.2. *As $z \longrightarrow 0$ we have that $|m_k| \sim |z|^{1-\frac{2k}{n+1}}$. In particular $|m_1| \sim |z|^{\frac{n-1}{n+1}}$ and $|m_n| \sim |z|^{-\frac{n-1}{n+1}}$.*

We will proceed by an induction argument on n. The induction argument will proceed by defining $\tilde{z} = z + m_1$, and so for this to be a small quantity we first show

Proposition 2.3. *In the limit $z \longrightarrow 0$ we have $m_1 = o(1)$.*

For this we define the saturated self-energy operator F by $F\mathbf{u} = |\mathbf{m}|S(|\mathbf{m}|\mathbf{u})$, meaning it has entries $F_{ij} = |m_i|s_{ij}|m_j|$. Then by (a slight modification of) [1, Proposition 7.2.9] we have that $\|F\| := \|F\|_{\ell^2 \to \ell^2} < 1$. The modification of [1, Proposition 7.2.9] is as follows. Since F^2 has strictly positive entries the Perron-Frobenius theorem for primitive matrices [2, Theorem 8.4.4] gives the existence of a Perron-Frobenius eigenvector. The remaining proof of [1, Proposition 7.2.9] is the same. Now, we may prove the proposition,

Proof of Proposition 2.3. Suppose for contradiction that $|m_1| > c$ for some sequence $z \longrightarrow 0$. Then

$$c|m_k| \leq |m_1||m_k| = F_{1k} \leq \|F\| \leq 1$$

and so every m_k is bounded. Hence by extracting a subsequence we may assume that all m_k converge as $z \longrightarrow 0$. Then

$$-1 = z\mathbf{m} + \mathbf{m}S\mathbf{m} \longrightarrow \mathbf{m}S\mathbf{m}$$

The bound $\|F\| \leq 1$ survives in the limit and thus

$$n = |\langle 1|\mathbf{m}S\mathbf{m}\rangle| = \left|\sum_{i,j} m_i s_{ij} m_j\right| \leq \sum_{i,j} |m_i| s_{ij} |m_j| = \langle 1|F1\rangle \leq n$$

We conclude that $\mathbf{m}S\mathbf{m} = \lambda^2|\mathbf{m}|S|\mathbf{m}| = \lambda^2 F1$ for some fixed $\lambda \in \mathbb{C}, |\lambda| = 1$. Thus $\mathbf{m} = \lambda|\mathbf{m}|$ and $-1 = \mathbf{m}S\mathbf{m} = \lambda^2 F1$. Since F only has non-negative entries this gives $\lambda^2 = -1$ and thus using the first and last equation we get $m_n m_1 = -1 = \sum_{j=1}^n m_1 m_j$. Hence,

$$0 = \sum_{j=1}^{n-1} m_1 m_j = -\sum_{j=1}^{n-1} |m_1||m_j|.$$

Thus $|m_1| = 0$ at $z = 0$. Contradiction. We conclude that $m_1 = o(1)$. □

We may now prove Proposition 2.2.

Proof of Proposition 2.2. This we prove by induction from $n-2$ to n. One may directly check the result for $n = 1, 2$. Hence, suppose the result is true for $n - 2$. Define $\tilde{z} = z + m_1$. Then by Proposition 2.3 we have $\tilde{z} = o(1)$ as $z \to 0$. The equations (2.1) now read

$$\begin{aligned}
\frac{-1}{m_1} &= z + m_1 + \cdots + m_n &&= \tilde{z} + m_2 + \ldots + m_n \\
\frac{-1}{m_2} &= z + m_1 + \cdots + m_{n-1} &&= \tilde{z} + m_2 + \ldots + m_{n-1} \\
&\vdots &&\vdots \\
\frac{-1}{m_{n-1}} &= z + m_1 + m_2 &&= \tilde{z} + m_2 \\
\frac{-1}{m_n} &= z + m_1 &&= \tilde{z}
\end{aligned}$$

Forgetting the first and last equation we thus have the equations

$$\begin{aligned}
\frac{-1}{m_2} &= \tilde{z} + m_2 + \ldots + m_{n-1} \\
&\vdots \\
\frac{-1}{m_{n-1}} &= \tilde{z} + m_2.
\end{aligned}$$

This system of equations is exactly of the same form as equations (2.1), only with $n-2$ variables m_k instead, and with small parameter $\tilde{z}$ instead of z. The induction hypothesis thus gives that $|m_k| \sim |\tilde{z}|^{1-\frac{2(k-1)}{n-1}}$ for all $2 \leq k \leq n-1$. This of course also holds for $k = n$.

We now show that $|z| \ll |m_1|$ so that $|\tilde{z}| \sim |m_1|$. Hence suppose for contradiction that $|m_1| = O(|z|)$. Considering the first and last equations we have

$$zm_1 + (m_1 + \ldots + m_n)m_1 = -1 = zm_n + m_1 m_n$$

Now, $|m_n| \gg 1 \gg |m_1|$ and so gathering the terms with z and writing only the highest order terms we get

$$zm_n = m_1 m_{n-1}(1 + o(1)).$$

Thus

$$|m_1| = \frac{|m_n|}{|m_{n-1}|}|z|(1+o(1)) \sim |\tilde{z}|^{\frac{-2}{n-1}}|z| = o(|z|).$$

Thus $|\tilde{z}| \sim |z|$ and so $|m_1| \sim |z|^{\frac{n+1}{n-1}}$. Then $|m_1 m_{n-1}| \sim |z|^{\frac{4}{n-1}}$ and so $zm_n = m_1 m_{n-1}(1+o(1)) = o(1)$. Similarly $m_1 m_n = o(1)$. Hence the last equation gives $-1 = zm_n + m_1 m_n = o(1)$. Contradiction. We conclude that $|z| \ll |m_1|$ and so $|\tilde{z}| \sim |m_1|$. Then we have $|m_k| \sim |m_1|^{1-\frac{2(k-1)}{n-1}}$.

Now, combining the first and last equations of (2.1) we get

$$zm_n + m_1 m_n = -1 = zm_1 + (m_1 + \ldots + m_n)m_1 = m_1 m_n + \Theta\left(|m_1|^{\frac{2}{n-1}}\right).$$

Thus we get that $z \sim |m_1|^{\frac{n+1}{n-1}}$, i.e. $|m_1| \sim |z|^{\frac{n-1}{n+1}}$ and so $|m_k| \sim |m_1|^{1-\frac{2(k-1)}{n-1}} \sim |z|^{1-\frac{2k}{n+1}}$ for all k. This finishes the proof. □

Now, we show that $\operatorname{Im} m_k \sim |m_k|$ so that indeed ρ has the desired behaviour. First, we have the formula.

Proposition 2.4. *The solution to equations (2.1) satisfies*

$$m_k = \left(\frac{z+m_1}{z}\right)^{k-1} m_1 \quad \textit{for all } k.$$

Proof. The equation for m_k reads

$$\frac{-1}{m_k} = z + m_1 + \ldots + m_{n+1-k}$$

Thus, combing the equation for m_k and m_{k+1} we get

$$\frac{1}{m_{k+1}} - \frac{1}{m_k} = m_{n+1-k}$$

Taking reciprocals and using the equation for m_{n+1-k} we thus get

$$\frac{m_k m_{k+1}}{m_{k+1} - m_k} = \frac{-1}{m_{n+1-k}} = z + m_1 + \ldots + m_{n+1-(n+1-k)} = z + m_1 + \ldots + m_k$$

Thus we get for $2 \leq k \leq n-1$ that

$$m_{k+1} = \frac{z + m_1 + \ldots + m_k}{z + m_1 + \ldots + m_{k-1}} m_k.$$

If $k = 1$ we instead get the formula

$$m_2 = \frac{z+m_1}{z} m_1.$$

Now, a simple induction argument gives the claimed formula. □

Now, we show that the imaginary parts of the m_k are of the same order as their norms. Let $\phi_k = m_k/|m_k|$ be the phase of m_k. Then we claim

Proposition 2.5. *As* $\operatorname{Im} z \ll \operatorname{Re} z \ll 1$ *then* $\phi_k = \exp\left(\frac{k\pi}{n+1} i\right) + o(1)$. *In particular* $\operatorname{Im}(m_k) \sim |m_k| \sim |z|^{1-\frac{2k}{n+1}}$.

Proof. The number z has phase $z/|z| = 1 + o(1)$. Proposition 2.4 now gives $m_k = m_1^k z^{1-k}(1 + o(1))$ since $|m_1| \gg |z|$. Hence we get for the phases that $\phi_k = \phi_1^k + o(1)$, in particular $\phi_n = \phi_1^n + o(1)$. Now, by the equation for m_n we have

$$-1 = zm_n + m_1 m_n = m_1 m_n + o(1) = \phi_1 \phi_n + o(1) = \phi_1^{n+1} + o(1).$$

We conclude that $\phi_1 = \exp\left(\frac{\pi}{n+1}i + \frac{2\pi l}{n+1}i\right) + o(1)$ for some $l = 0, 1, \ldots, n$. We want to show that $l = 0$. For any k we have $\operatorname{Im} m_k > 0$. This means that $\frac{\pi}{n+1}(1 + 2l)k \in (0, \pi) + 2\pi\mathbb{Z}$ for all k, i.e. $(1 + 2l)k \in (0, n + 1) + 2(n + 1)\mathbb{Z}$.

Taking $k = 1$ in this we get $2l + 1 < n + 1$, since $2l + 1 < 2(n + 1)$. Hence $l < n/2$. Then taking $k = 2$ we get $2(2l + 1) < n + 1$, since $2(2l + 1) < 2(n + 1)$. Hence $l < \frac{n-1}{4}$. Continuing in this fashion we conclude that $l < \frac{n+1-k}{2k}$ for all $k = 1, \ldots, n$. We conclude that $l = 0$. This shows the desired. □

Since ρ is symmetric this gives the desired asymptotic behaviour for ρ.

3 Variance varies between blocks

We now consider the case where the variance is allowed to vary between the blocks, but stays constant inside each block. That is, the variances are given by the matrix

$$S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1(n-1)} & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2(n-1)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{(n-1)1} & s_{(n-1)2} & \cdots & 0 & 0 \\ s_{n1} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where each s_{ij} has all the same entries (which might differ for different i, j). Again, we may treat both each $s_{ij} \in \mathbb{R}^{N\times N}$ and each $\mathbf{m}_k \in \mathbb{H}^N$ as numbers $s_{ij} \in \mathbb{R}$ and $m_k \in \mathbb{H}$. Note that as there are only finitely many s_{ij} (independent of N) we have $c < s_{ij} < C$ if $i + j - 1 \leq n$ and $s_{ij} = 0$ otherwise. The vector Dyson equation is then again

$$\frac{-1}{\mathbf{m}} = z + S\mathbf{m} \qquad \text{i.e.} \qquad \frac{-1}{m_k} = z + \sum_{j=1}^{n+1-k} s_{kj} m_j, \qquad k = 1, \ldots, n,$$

where $z \in \mathbb{H}$ and $\mathbf{m} \in \mathbb{H}^n$. We prove that, in this case Conjecture 1.1 holds.

Theorem 3.1. *In the limit $E \to 0$ we have that $\rho(E) \sim |E|^{-\frac{n-1}{n+1}}$.*

We discuss how the argument above generalises, where it breaks down and how to fix it.

Most of the proof works equally well in this case. The only problem is the induction step. The construction $\tilde{z} = z + m_1$ is not the correct one to make: The equations are

$$\frac{-1}{m_k} = z + s_{k1} m_1 + \ldots + s_{k(n+1-k)} m_{n+1-k}, \qquad k = 1, \ldots, n.$$

Thus, in order to absorb the m_1-terms into the small parameter z we would need to define $\tilde{z} = z + (s_{11} m_1, \ldots, s_{n1} m_1)$. This is however not a number, so something needs to be done for the induction argument to work. We will do as follows.

Treat instead $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{H}^n$ as a vector instead of the number $z \in \mathbb{H}$ and consider the equations

$$\frac{-1}{m_k} = z_k + s_{k1} m_1 + \ldots + s_{k(n+1-k)} m_{n+1-k}, \qquad k = 1, \ldots, n.$$

We are really interested in the case $\mathbf{z} = (z, z, \ldots, z) \in \mathbb{H}^n$, since this corresponds to the original equations. Hence a natural condition to require on $\mathbf{z}$ is that all components are of the same order, $|z_j| \sim |z_k|$ for all j, k. We will denote this order by $|z|$, i.e. $|z_k| \sim |z|$ for all k. Proposition 2.3 works equally well in this case, where $\mathbf{z}$ is a vector. The proof is the same. In fact, for this result, we don't need to impose the condition that all components of $\mathbf{z}$ have similar size.

Proposition 3.2. *In the limit* $\mathbf{z} \to 0$ *we have* $m_1 = o(1)$.

For the induction argument Proposition 2.2 we have mostly the same proof, only we need to check that $\tilde{\mathbf{z}} = \mathbf{z} + (s_{11}m_1, \ldots, s_{n1}m_1)$ has similar size components in order for the induction step to work.

Proposition 3.3. *Let* $\mathbf{z} \in \mathbb{H}^n$ *with* $|z_k| \sim |z|$ *for all* k. *As* $\mathbf{z} \to 0$ *we have that* $|m_k| \sim |z|^{1-\frac{2k}{n+1}}$. *In particular* $|m_1| \sim |z|^{\frac{n-1}{n+1}}$ *and* $|m_n| \sim |z|^{-\frac{n-1}{n+1}}$.

Proof. The proof is mostly the same as above, i.e. by induction on n. For the induction to work however, we need to check that the small parameter $\tilde{\mathbf{z}} = \mathbf{z} + (s_{11}m_1, \ldots, s_{n1}m_1)$ has similar size components. Hence suppose not. Then z_k and $s_{k1}m_1$ must have almost opposite phases for some k. As $\operatorname{Im} \mathbf{z} > 0$ and $\operatorname{Im} m_1 > 0$, this implies, that $\frac{\operatorname{Im} m_1}{|m_1|}$ vanishes. Then the first equation gives us

$$\frac{\operatorname{Im} m_1}{|m_1|} = \eta_1 |m_1| + |m_1| s_{11} \operatorname{Im} m_1 + \ldots + |m_1| s_{1n} \operatorname{Im} m_n \geq F_{1n} \frac{\operatorname{Im} m_n}{|m_n|}$$

and so $\frac{\operatorname{Im} m_n}{|m_n|}$ also vanishes. Now we show that $z_n m_n$ vanishes.

The last equation gives

$$\frac{\operatorname{Im} m_n}{|m_n|} = \eta_n |m_n| + F_{n1} \frac{\operatorname{Im} m_1}{|m_1|}$$

and thus $\eta_n |m_n|$ vanishes. If $E_n = O(\eta_n)$ then $z_n m_n$ vanishes, and so we may assume that $\eta_n = o(E_n)$. Also, by the symmetry of the equations $\mathbf{z} \to -\bar{\mathbf{z}}, \mathbf{m} \to -\bar{\mathbf{m}}$ it suffices to consider $E_n > 0$. Now, the equation for m_n reads $-1 = z_n m_n + m_n s_{n1} m_1$ and so $|1 + z_n m_n| = |m_n s_{n1} m_1| = F_{n1} \leq 1$. Hence $z_n m_n \in B(-1, 1)$, the ball of radius 1 centered at -1. In particular $\operatorname{Im}(z_n m_n) < 0$. So

$$0 > \frac{\operatorname{Im}(z_n m_n)}{|m_n|} = E_n \frac{\operatorname{Im} m_n}{|m_n|} + \eta_n \frac{\operatorname{Re} m_n}{|m_n|}.$$

Since $E_n, \eta_n, \operatorname{Im} m_n > 0$ we conclude that $\operatorname{Re} m_n < 0$ and thus

$$\frac{\operatorname{Re} m_n}{|m_n|} = -1 + o(1).$$

We conclude that $\frac{\operatorname{Im} m_n}{|m_n|} = O\left(\frac{\eta_n}{E_n}\right)$. Let $\arg$ denote the argument function taking values in $(0, 2\pi)$. Then

$$\arg(z_n m_n) = \arg(z_n) + \arg(m_n) = \left(O\left(\frac{\eta_n}{E_n}\right)\right) + \left(\pi - O\left(\frac{\operatorname{Im} m_n}{|m_n|}\right)\right) = \pi + O\left(\frac{\eta_n}{E_n}\right).$$

Since $z_n m_n \in B(-1, 1)$ we thus conclude that $|z_n m_n| = o(1)$. Now we show that $|m_1 m_n| > c$ as $\mathbf{z} \to 0$.

Rearranging the equations for the real values we have $\mathbf{E}|\mathbf{m}| = (1 + F)\frac{\operatorname{Re} \mathbf{m}}{|\mathbf{m}|}$. Thus

$$o(1) = E_n |m_n| = \left\langle e_n \middle| (1 + F) \frac{\operatorname{Re} \mathbf{m}}{|\mathbf{m}|} \right\rangle = -\frac{\operatorname{Re} m_n}{|m_n|} - F_{n1} \frac{\operatorname{Re} m_1}{|m_1|}.$$

Thus $F_{n1} \to 1$. In particular $|m_1||m_n| > \frac{1}{2s_{n1}} = c$ for $\mathbf{z}$ small enough. Thus $|z_n||m_n| \ll 1 \sim |m_1||m_n|$ and so $|z| \sim |z_n| \ll |m_1|$. Then $|\tilde{z}_k| = |z_k + s_{k1} m_1| \sim |m_1|$ all have the same order. Contradiction. We conclude that all components of $\tilde{\mathbf{z}}$ have the same order.

The remaining parts of the proof is exactly the same as in the proof of Proposition 2.2. □

We may again now just consider $z \in \mathbb{H}$ a number, since the only reason we needed to consider $\mathbf{z} \in \mathbb{H}^n$ was the technical argument in Proposition 3.3 above. The formula Proposition 2.4 is also not true in general, but it is true approximately.

Proposition 3.4. *As $z \to 0$ then $m_k = c_k z^{1-k} m_1^k\,(1 + o(1))$ for some (explicit) constants $c_k > 0$.*

The proof of Proposition 3.4 is similar to the proof of Proposition 2.4 only we need to take care of the error terms since we in general don't have much cancellation of the terms.

Proof. Define $\lambda = |z|^{2/(n+1)}$. Then, expanding to second order we have

$$\frac{1}{m_{k+1}} - \frac{1}{m_k} = s_{k(n+1-k)} m_{n+1-k} \left(1 + \frac{s_{k(n-k)} - s_{(k+1)(n-k)}}{s_{k(n+1-k)}} \frac{m_{n-k}}{m_{n+1-k}} + O(\lambda^2)\right)$$

Taking the reciprocal and using the equation for m_{n+1-k} we thus get

$$\frac{m_{k+1} m_k}{m_{k+1} - m_k} = m_k + \frac{s_{(n+1-k)(k-1)}}{s_{(n+1-k)k}} m_{k-1} + \frac{s_{(k+1)(n-k)} - s_{k(n-k)}}{s_{k(n+1-k)}} \frac{m_k m_{n-k}}{m_{n+1-k}} + O(|m_k|\lambda^2).$$

Thus

$$-m_k^2 + \frac{s_{(n+1-k)(k-1)}}{s_{(n+1-k)k}} m_{k+1} m_{k-1} + \frac{s_{(k+1)(n-k)} - s_{k(n-k)}}{s_{k(n+1-k)}} \frac{m_k m_{k+1} m_{n-k}}{m_{n+1-k}} = O(|m_k|^2 \lambda).$$

Now we use that $m_k s_{k(n+1-k)} m_{n+1-k} = -1 + O(\lambda)$, which follows from the equation for m_k. With this we get

$$-1 + \frac{s_{(n+1-k)(k-1)}}{s_{(n+1-k)k}} \frac{m_{k+1} m_{k-1}}{m_k^2} + \left(1 - \frac{s_{k(n-k)}}{s_{(k+1)(n-k)}}\right) = O(\lambda)$$

Hence $m_{k+1} = \tilde{c}_{k+1} m_k^2 m_{k-1}^{-1}(1 + O(\lambda))$ for $\tilde{c}_{k+1} = \frac{s_{(n+1-k)k} s_{k(n-k)}}{s_{(n+1-k)(k-1)} s_{(k+1)(n-k)}} > 0$.

For the case where $k = 1$ everything is the same only with $O(\lambda)$ replaced by $o(1)$ and $m_0 = z$. Thus, by induction we conclude the desired formula. ☐

With this Proposition 2.5 carries over immediately, i.e.

Proposition 3.5. *As $\operatorname{Im} z \ll \operatorname{Re} z \ll 1$ then $\phi_k = \exp\left(\frac{k\pi}{n+1} i\right) + o(1)$. In particular $\operatorname{Im}(m_k) \sim |m_k| \sim |z|^{1-\frac{2k}{n+1}}$.*

This shows the desired on ρ.

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