FREE ENERGY IN THE SHERRINGTON-KIRKPATRICK MODEL

DARIO FELICIANGELI

MAY 25, 2018

1. INTRODUCTION

The Sherrington-Kirkpatrick model (SK model) was introduced by Sherrington and Kirkpatrick in 1975 in [5] to describe spin glasses: disordered magnetic alloys which exhibit an unusual physical behaviour. In this article the authors already proposed an heuristic solution to the problem which, although, turned out to be incorrect. Some years later, towards the end of the seventies, it was Giorgio Parisi ([7], [8]) to discover the right formula for the limit of the free energy in the SK model, which, since then, has gone under the name of Parisi Ansatz or Parisi Formula. Althought this formula was believed to be correct almost immediately, it took the mathematical and physical community thirty years of work (with contributions from Guerra, Aizenman, Ruelle and many others) before Talagrand finally managed to put together all the known theory to show, in 2006 (see [10]), the validity of the Parisi Ansatz.

In this short report we will introduce the model and give an overview of some results related to the limit of the free energy. In particular, we will first treat the high temperature regime, for which the existence and the precise value of the limit of the free energy was already rigorously proved in 1987 (see [1]), secondly we will move on to the discussion of the Guerra-Toninelli theorem (see [4]), which elegantly settles the matter of the convergence of the free energy in the general case and was only proved in 2002.

Our dissertation will be basically based on two references: for the high temperature regime we considered mainly [1] by M. Aizenman, J.L. Lebowitz and D. Ruelle, whereas for the Guerra-Toninelli theorem we followed the approach developed in [6], a remarkable book by D. Panchenko which is a self-contained overview of the known results on the SK model.

2. THE SHERRINGTON-KIRKPATRICK MODEL

To introduce the SK model, we consider N spins, i.e. a collection of N variables $\sigma_1, ..., \sigma_N$ which can take values in $\{-1, 1\}$. We will call spin configuration any element σ of $\Sigma_N := \{-1, 1\}^N$, on this set we define the following random Hamiltonian:

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{1\leq i<j\leq N} J_{ij}\sigma_i\sigma_j. \tag{2.1}$$

The coefficients J_{ij}, in the most general setting, are i.i.d. random variables with mean 0 and variance J^2, symmetricly distributed w.r.t. 0. The scaling factor $\frac{1}{\sqrt{N}}$ comes into play to assure that the total energy of the system scales like N, i.e. the energy per spin is of order one. We observe that considering an Hamiltonian depending on J_{ij} also for $i \leq j$ would make no difference both from the mathematical and the physical point of view, but our choice is to stick to definition (2.1) in the following.

The two main features of this model are the randomness in the Hamiltonian and the fact that the interaction between spins is of mean-field type (i.e. there is interaction between all spins). Generally speaking, in statistical mechanics, mean-field models are usually easier approximations of local models (which are much harder, if not impossible, to solve), the key point being that in the mean-field regime it is often possible to perform explicit computations. This general perspective also applies to the SK-model (still a very hard model to solve), which is the mean-field approximation of the Edwards-Anderson (EA) model, in which the interaction is still random but only between neighbouring spins. At the same time, it is possible to see SK as a randomization of the Curie-Weiss model (deterministic mean-field type interaction), which in turn is the mean-field approximation of the Ising model (deterministic interaction only between neighbours).

We will denote by β the inverse temperature (i.e. $\beta = 1/T$) and by $Z_N(\beta)$ the partition function at temperature $1/\beta$:

$$Z_N(\beta) = \sum_{\sigma \in \Sigma_N} \exp \beta H_N(\sigma).$$

A very important quantity is the free energy per spin, defined by:

$$F_N(\beta) = \frac{1}{N} \mathbb{E} \log Z_N(\beta).$$

We observe that the free energy can be seen as a low temperature approximation of the renormalized expected value of the maximum of the energy over all the spin configurations. Indeed it is an easy computation to show that:

$$\frac{1}{N} \mathbb{E} \max_{\sigma \in \Sigma_N} H_N(\sigma) \leq \frac{F_N(\beta)}{\beta} \leq \frac{\log(2)}{\beta} + \frac{1}{N} \mathbb{E} \max_{\sigma \in \Sigma_N} H_N(\sigma). \tag{2.2}$$

We also define the Gibbs measure, a random probability measure on Σ_N, by:

$$G(\sigma) = \frac{\exp \beta H_N(\sigma)}{Z_N(\beta)}. \tag{2.3}$$

It is trivial to observe that the Gibbs measure is concentrated on the configurations for which $H_N(\sigma)$ is bigger and hence having a good understanding of its asymptotical structure would also yield results for the limit of the free energy. Such analysis is indeed essential to compute the limit of the free energy in the low temperature regime (for a complete review we refer to [6]), but we will attempt this challenge only in the high temperature regime, where everything is much easier, whereas, in the low temperature regime, we will settle for the proof of the existence of the limit of $F_N(\beta)$.

A final remark: the existence of the limit of the free energy, $F(\beta) := \lim_{N\to\infty} F_N(\beta)$, can also be used to show the existence of $\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \max_{\sigma \in \Sigma_N} H_N(\sigma)$. Indeed, by applying Hölder's inequality we get, for every $t \geq 1$:

$$\beta^{-1} \left(F_N(\beta) - \log 2 \right) = \frac{1}{N\beta} \mathbb{E} \log \frac{1}{2^N} \sum_{\sigma \in \Sigma_N} \exp \beta H_N(\sigma)$$
$$\leq \frac{1}{Nt\beta} \mathbb{E} \log \frac{1}{2^N} \sum_{\sigma \in \Sigma_N} \exp t\beta H_N(\sigma).$$

This implies that $\beta \mapsto \beta^{-1} \left(F_N(\beta) - \log(2) \right)$ is monotonically increasing. Moreover, if we have convergence of the free energy, passing to the limit $N \to \infty$ we also get that $\beta^{-1} \left(F(\beta) - \log(2) \right)$ is monotone increasing in β. This, in turn, implies that $\frac{F(\beta)}{\beta}$ admits limit for $\beta \to \infty$, finally, putting this together with equation (2.2), we arrive at:

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E} \max_{\sigma\in\Sigma_N} H_N(\sigma) = \lim_{\beta\to\infty} \frac{F(\beta)}{\beta}.$$

3. The High-Temperature Regime

In this section we will consider the interaction J_{ij} in (2.1) to be i.i.d. random variables with mean zero, variance J^2, even distribution and such that the moment-generating function of J_{ij} exists in a neighbourhood of the origin. We study the case of high temperature, in particular we consider $\beta J < 1$.

The first step of our analysis is to compute the expected value of the partition function, using that J_{ij} are i.i.d. and have symmetric distribution:

$$\mathbb{E}Z_N(\beta) = \sum_{\sigma\in\Sigma_N} \mathbb{E}\prod_{i<j} \exp\frac{\beta J_{ij}}{\sqrt{N}}\sigma_i\sigma_j = \sum_{\sigma\in\Sigma_N}\prod_{i<j}\mathbb{E}\exp\frac{\beta J_{ij}}{\sqrt{N}}\sigma_i\sigma_j =$$
$$= \sum_{\sigma\in\Sigma_N}\prod_{i<j}\mathbb{E}\left(\frac{1}{2}\exp\frac{\beta J_{ij}}{\sqrt{N}}\sigma_i\sigma_j + \frac{1}{2}\exp\frac{-\beta J_{ij}}{\sqrt{N}}\sigma_i\sigma_j\right) = 2^N\prod_{i<j}\mathbb{E}\left(\cosh\frac{\beta J_{ij}}{\sqrt{N}}\right).$$

Moreover, using Taylor expansion for $\cosh$ and $\log$ (which is legit, at least for N big enoug, by existence of the moment generating function in a neighbourhood of the origin), we get:

$$\log \mathbb{E}Z_N(\beta) = N\left(\log 2 + \frac{1}{4}\beta^2 J^2\right) + \frac{\beta^4}{2\cdot 4!}(\mathbb{E}J_{12}^4 - 3J^4) + O(1/N). \tag{3.1}$$

This implies that we can precisely compute $\lim \frac{1}{N}\log \mathbb{E}Z_N(\beta)$, a quantity which is indeed similar to the free energy $F_N(\beta)$, but not quite. The main feature of the high temperature regime is that we can actually interchange the expected value and the logarithm, i.e. the following theorem, which is the main result of this section, holds.

Theorem 3.1. *If $\beta J < 1$, then:*

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N(\beta) = \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_N(\beta),$$

thus:

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N(\beta) = \log 2 + \frac{1}{4}\beta^2 J^2 =: Q(\beta).$$

To see why theorem 3.1 holds, we begin by decomposing $Z_N(\beta)$ in the following way:

$$Z_N(\beta) = \sum_{\sigma\in\Sigma_N}\prod_{i<j}\exp\frac{\beta J_{ij}}{\sqrt{N}}\sigma_i\sigma_j = \sum_{\sigma\in\Sigma_N}\prod_{i<j}\left(\cosh\frac{\beta J_{ij}}{\sqrt{N}} + \sigma_i\sigma_j\sinh\frac{\beta J_{ij}}{\sqrt{N}}\right)$$
$$= \sum_{\sigma\in\Sigma_N}\prod_{i<j}\left(\cosh\frac{\beta J_{ij}}{\sqrt{N}}\right)\left(1+\sigma_i\sigma_j\tanh\frac{\beta J_{ij}}{\sqrt{N}}\right) = 2^N\prod_{i<j}\left(\cosh\frac{\beta J_{ij}}{\sqrt{N}}\right)\widehat{Z_N(\beta)}.$$

Here

$$\widehat{Z_N(\beta)} := \frac{1}{2^N}\sum_{\sigma\in\Sigma_N}\prod_{i<j}(1+\sigma_i\sigma_j\tanh(\beta J_{ij}/\sqrt{N})).$$

At any temperature, using the previous computations (Equation (3.1)) and the central limit theorem, we have:

$$\frac{Z_N(\beta)}{\widehat{Z_N(\beta)}} = 2^N \exp\left[\sum_{ij} \log\left(\cosh\frac{\beta J_{ij}}{\sqrt{N}}\right)\right] = 2^N \exp\left[\sum_{ij}\left(\frac{\beta^2 J_{ij}^2}{2N} - \frac{2\beta^4 J_{ij}^4}{4!N^2}\right) + O\left(\frac{1}{N}\right)\right]$$

$$=2^N \exp\left[\frac{N}{2}\beta^2 J^2 + \frac{1}{2N}\sum_{ij}\beta^2(J_{ij}^2 - J^2) - \frac{\beta^4}{12}\mathbb{E}J_{12}^4 + O\left(\frac{1}{N}\right)\right]$$

$$=\mathbb{E}Z_N(\beta)\exp\left[\frac{1}{2N}\sum_{ij}\beta^2(J_{ij}^2 - J^2) - \frac{\beta^4}{8}(\mathbb{E}J_{12}^4 - J^4) + O\left(\frac{1}{N}\right)\right]$$

$$=\mathbb{E}Z_N(\beta)\exp\left(\tilde{v}_N - \frac{1}{2}\mathbb{E}\tilde{v}_N^2 + O\left(\frac{1}{N}\right)\right),$$

where

$$\tilde{v}_N := \sum_{ij}\frac{\beta^2}{2N}(J_{ij}^2 - J^2)$$

is a sequence of random variables, which, by the CLT, converges in distribution to a normal r.v. v with mean zero and variance $\frac{\beta^4}{8}\left(\mathbb{E}J_{12}^4 - J^4\right)$. This implies that:

$$\frac{1}{N}\log Z_N(\beta) = \frac{1}{N}\left(\log \mathbb{E}Z_N(\beta) + \log\widehat{Z_N(\beta)} + \tilde{v}_N - \frac{1}{2}\mathbb{E}\tilde{v}_N^2 + O\left(\frac{1}{N}\right)\right), \tag{3.2}$$

where $\frac{1}{N}\log\widehat{Z_N(\beta)}$ is the only term left to study to complete the proof of theorem 3.1, since $\tilde{v}_N - \frac{1}{2}\mathbb{E}\tilde{v}_N^2$ is is asymptotically normal of order $O(1)$ and thus $\frac{1}{N}(\tilde{v}_N - \frac{1}{2}\mathbb{E}\tilde{v}_N^2)$ vanishes in the limit.

Indeed, we will prove that, as a special feature of the high temperature regime, we have:

$$\widehat{Z_N(\beta)} \xrightarrow{\mathscr{D}} u \quad \text{as} \quad N \to \infty, \tag{3.3}$$

for some log-normal distribution u and hence also $\log\widehat{Z_N(\beta)}$ is of order $O(1)$, yielding:

$$\frac{1}{N}\log Z_N(\beta) \xrightarrow{\mathscr{D}} Q(\beta) = \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_N(\beta).$$

Generally, convergence in distribution does not imply convergence of the first moment, but, in this case, it does, since the following bounds for $\frac{1}{N}\log Z_N(\beta)$ hold:

- $\frac{1}{N}\log Z_N(\beta) \geq \log 2$,
- $\mathbb{P}\left(\frac{1}{N}\log Z_N(\beta) \geq Q_N(\beta) + \varepsilon\right) \leq e^{-\varepsilon N}$,

where we have denoted by $Q_N(\beta)$ the finite volume approximant of $Q(\beta)$. The first inequality comes from an easy direct computation and the second one from Markov inequality and (3.1). Hence, as claimed before, this strategy shows that theorem 3.1 holds, concluding the study of the high temperature regime.

To make this rigorous, we need to prove (3.3), which is exactly what the next theorem is about. We observe that all the other facts discussed so far are true at any temperature and relatively easy to show, hence, in some sense, the whole problem of studying the high temperature regime (theorem 3.1), reduces, and is equivalent, to the following theorem.

Theorem 3.2. *For $\beta J < 1$, we have:*

$$\widehat{Z_N(\beta)} \xrightarrow{\mathscr{D}} \tilde{u} := \exp\left(u - \frac{1}{2}\mathbb{E}u^2\right), \tag{3.4}$$

where u is a Gaussian r.v. with variance:

$$\mathbb{E}u^2 = -\frac{1}{2}\left[\log(1-\beta^2J^2) + \beta^2J^2 + \frac{1}{2}\beta^4J^4\right] = \sum_{k=3}^{\infty}\frac{1}{2k}(\beta^2J^2)^k. \tag{3.5}$$

Moreover, we also have:

$$\frac{Z_N(\beta)}{\mathbb{E}Z_N(\beta)} = \widehat{Z_N(\beta)}\exp\left(\tilde{v}_N - \frac{1}{2}\mathbb{E}\tilde{v}_N^2 + O\left(\frac{1}{N}\right)\right) \xrightarrow{\mathscr{D}} \exp\left(w - \frac{1}{2}\mathbb{E}w^2\right), \tag{3.6}$$

for w Gaussian random variable with variance:

$$\mathbb{E}w^2 = -\frac{1}{2}\left[\log(1-\beta^2J^2) + \beta^2J^2 - \frac{\beta^4\left(\mathbb{E}J_{12}^4 - 3(\mathbb{E}J_{12}^2)^2\right)}{4}\right].$$

Sketch of the proof. We will not prove statement (3.6) but we are mentioning it for sake of completeness. Indeed, it completes our knowledge of the limit behaviour, as $N\to\infty$, of the two fluctuations $\widehat{Z_N(\beta)}$ and $\exp\left(\tilde{v}_N - \frac{1}{2}\mathbb{E}\tilde{v}_N^2 + O\left(\frac{1}{N}\right)\right)$, stating that they add up as a pair of independent Gaussian.

In the direction of showing (3.4) and (3.5) (which is just a by-product of the proof of (3.4)), the first key passage is to rewrite $\widehat{Z_N(\beta)}$ in a more manageable way:

$$\widehat{Z_N(\beta)} = \frac{1}{2^N}\sum_{\sigma\in\Sigma_N}\left(1 + \sum_{i<j}\sigma_i\sigma_j\tanh\left(\frac{\beta J_{ij}}{\sqrt{N}}\right) + \sum_{\substack{i<j,k\\k<l}}\sigma_i\sigma_j\sigma_k\sigma_l\tanh\left(\frac{\beta J_{ij}}{\sqrt{N}}\right)\tanh\left(\frac{\beta J_{kl}}{\sqrt{N}}\right) + \dots\right).$$

The sums in the previous equation only run through distinct couples, as they come from the expansion of a product with all distinct factors. Now we can pull the average on Σ_N inside the parenthesis and observe that only terms in which each σ_t appears an even number of times survive. This observation allows us to write, very compactly:

$$\widehat{Z_N(\beta)} = \sum_{\gamma\in\Gamma} w(\gamma),$$

where we define Γ to be the set of all simple and closed graphs on $\{1,2,...,N\}$, i.e. graphs without repetition of edges and such that there are no vertices with odd degree, and:

$$w(\gamma) := \prod_{e\in E(\gamma)}\tanh\left(\frac{\beta J_e}{\sqrt{N}}\right).$$

Now we can observe the following: if we consider the L^2 norm of $\widehat{Z_N(\beta)}$, then we have:

$$\mathbb{E}\left(\sum_{\gamma\in\Gamma} w(\gamma)\right)^2 = \sum_{\gamma\in\Gamma}\mathbb{E}(w(\gamma)^2). \tag{3.7}$$

Indeed, if $\gamma\neq\gamma'$ then there is at least one edge that appears only in γ or in γ' and this implies, since $\tanh$ is odd and J_{ij} are independent and symmetric w.r.t. the origin, that the expectation of $w(\gamma)w(\gamma')$ will be zero. Hence, $w(\gamma)$'s for different γ's are orthogonal w.r.t. the L^2 product and (3.7) holds. The sum now is running over graphs in Γ but, since J_{ij} are independent i.i.d random variables, obviously the contribution to the L^2 norm of $\widehat{Z_N(\beta)}$ of every single γ does not depend on the particular labeling of vertices in γ. Thus, it is clear that the problem of computing the L^2 norm of $\widehat{Z_N(\beta)}$ actually consists of two parts: understanding the weight associated to each class of graphs in Γ (i.e. up to labelings of the vertices) and then computing the number of times that a realization of each class appears in Γ. Let $\gamma \in \Gamma$ then we can immediately get:

$$\mathbb{E}(w(\gamma)^2) = \mathbb{E} \prod_{e \in E(\gamma)} \left[\tanh\left(\frac{\beta J_e}{\sqrt{N}}\right)\right]^2 \leq (\beta^2 J^2)^{|E(\gamma)|} \frac{1}{N^{|E(\gamma)|}}, \tag{3.8}$$

where we denote by $E(\gamma)$ the set of edges in γ and by $V(\gamma)$ the set of vertices in γ. On the other hand, it is immediate to compute that the number of graphs in the same class of γ is proportional to $N^{|V(\gamma)|}$. Putting these two facts together, we get that each class' contribution to the L^2 norm of $\widehat{Z_N(\beta)}$ is of order $O\left(N^{|V(\gamma)|-|E(\gamma)|}\right)$. Hence, since we are already summing only over closed and simple graphs, the only graphs whose contributions are of order $O(1)$ are simple loops or unions of pairwise disjoint simple loops (for which $|V(\gamma)| - |E(\gamma)| = 0$), whereas the other graphs have contributions of order at most $O\left(\frac{1}{N}\right)$ and will then vanish (if considered individually) in the limit $N \to \infty$. This is telling us, at least heuristically, that graphs which are not simple loops or unions of pairwise disjoint simple loops play a smaller role, if any, in the limit behaviour of $\widehat{Z_N(\beta)}$ (vanishing L^2-weight implies that they are not relevant for convergence in probability or in distribution). This can be shown rigorously and it is even possible to show that the terms coming from unions of more than one simple loop are irrelevant: studying the convergence in distribution of $\widehat{Z_N(\beta)}$ is equivalent to just study the convergence in distribution of:

$$V_N(\beta) = \sum_{\gamma \text{ simple loop}} w(\gamma).$$

We stress that the proof of this fact (which we will not give for brevity, the interested reader is invited to refer to [1]) is still highly non trivial and that actually, with the previous discussion, we were only trying to convince the reader that this approximation step is at least reasonable. In any way, we will take this fact for granted and immediately jump to discuss the convergence of $V_N(\beta)$.

Our strategy will be the following: for every $\varepsilon > 0$ we will try to construct a sequence of random variables $V_N^\varepsilon(\beta)$ which is an ε-approximation of $V_N(\beta)$ in the L^2-sense uniformly in N and which converges in distribution as $N \to \infty$. If we can manage to do this, then by a well known criterion for convergence in distribution (see Lemma 3.3 at the end of the section), we will have that the following quantities exist and coincide:

$$\mathscr{D}\text{-}\lim_N V_N(\beta) = \mathscr{D}\text{-}\lim_\varepsilon \; \mathscr{D}\text{-}\lim_N V_N^\varepsilon(\beta).$$

In some sense, we have a natural candidate for the approximation of $V_N(\beta)$, since we can write:

$$\sum_{\gamma \text{ simple loop}} w(\gamma) = \sum_{|\gamma| \leq k} w(\gamma) + \sum_{|\gamma| > k} w(\gamma) \equiv V_N^{\leq k}(\beta) + V_N^{>k}(\beta).$$

Indeed, using that $\mathbb{E} \tanh(\beta J_e/\sqrt{N}) = 0$ since $\tanh$ is odd and J_e is symmetrically distributed w.r.t. the origin and also using that all γ are simple, we can get the orthogonality of the weights $w(\gamma)$ w.r.t. to the L^2-product, which implies, recalling that the J_{ij} are i.i.d and doing some computations:

$$\mathbb{E}\Big(\sum_{|\gamma|=k} w(\gamma)\Big)^2 = \mathbb{E} \sum_{|\gamma|=k} w(\gamma)^2 = \frac{N!}{(N-k)!2k} \left[\mathbb{E} \tanh\left(\frac{\beta J_{12}}{\sqrt{N}}\right)^2\right]^k \nearrow \frac{(\beta^2 J^2)^k}{2k}.$$

This immediately implies (we stress the importance of the hypothesis $\beta J < 1$) that $||V_N^{>k}(\beta)||_2 < \varepsilon_k$, uniformly in N, with $\varepsilon_k \to 0$ as $k \to \infty$ (and hence $V_N^{\leq k}(\beta)$ approximates $V_N(\beta)$ in the sense of the previous criterion) and that:

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbb{E}\left[V_N^{\le k}(\beta)^2\right]=\mathbb{E}u^2. \tag{3.9}$$

If we can now manage to show the same kind of limiting behaviour also for the higher moments of $V_N^{\le k}(\beta)$, we will be done, since convergence of the moments implies convergence in distribution if the limit moments are nice enough (which is for sure the case for the moments of a Gaussian).

Thus we need to show that, for every $2<t\in\mathbb{N}$ and every k:

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbb{E}\left[V_N^{\le k}(\beta)^t\right]=\mathbb{E}u^t=|\{\text{pairings of }\{1,...,t\}\}|\cdot\left(\mathbb{E}u^2\right)^{t/2}, \tag{3.10}$$

where in the last passage we have used Wick's law for the moments of a Gaussian. We observe that, by (3.9), to get (3.10) it is sufficient to show that, for every $2<t\in\mathbb{N}$ and for every k:

$$R_{k,t}(N)\equiv\mathbb{E}\left[V_N^{\le k}(\beta)^t\right]-|\{\text{pairings of }\{1,...,t\}\}|\cdot\left[\mathbb{E}\left(V_N^{\le k}(\beta)^2\right)\right]^{t/2}\xrightarrow{N\to\infty}0. \tag{3.11}$$

Hence we have fully reduced our problem of showing (3.5) and (3.4), to claim (3.11).

We write:

$$R_{k,t}(N)=\sum_{\substack{\gamma_1,...,\gamma_t\\|\gamma|\le k}}\mathbb{E}w(\gamma_1)...w(\gamma_t)-\sum_{\substack{\text{pairings of}\\\{1,...,t\}}}\sum_{\substack{t/2-\text{tuples of}\\\text{couples }(\gamma_i,\gamma_j)\\|\gamma|\le k}}\mathbb{E}w(\gamma_{i_1},\gamma_{j_1})...\mathbb{E}w(\gamma_{i_{n/2}},\gamma_{j_{n/2}}).$$

Each term in the sums above can be thought as a multigraph Υ for which each edge ij has multiplicity n_{ij} given by the number of simple loops in which it appears. Even if this correspondence is well defined, it is not invertible because the same multigraph could correspond to different terms. Some of these multigraphs are special: the ones which consist of pairwise disjoint double loops (not sharing vertices nor edges). Each such graph Υ appears exactly $\binom{t}{2,...,2}$ times in the first sum and $(t/2)!$ times in the inner part of the second sum (it appears more times but when the simple loops are not coupled with their twin then the whole term is zero) and thus its contribution to $R_{k,t}(N)$ is zero. As we will see, these multigraphs would be the only ones with contributions of order $O(1)$ in N to $R_{k,t}(N)$. Getting rid, thanks to cancelation, of these troublesome terms is then very important, since now it suffices to show that all the other multigraphs have contributions of order $O\left(\frac{1}{N}\right)$ that will vanish in the limit for large N. Indeed, if we fix one t-tuple in the first sum, we have that the absolute value of its contribution to $R_{k,t}(N)$ is bounded by the following function of the corresponding multigraph Υ:

$$\Theta(\Upsilon)=\prod_{i<j}\mathbb{E}\left[\left(\frac{\beta^2J_{12}^2}{N}\right)^{n_{ij}/2}\right]\le\left[\beta\left(\mathbb{E}|J_{12}|^t\right)^{1/t}\right]^{|\Upsilon|}\cdot N^{-|\Upsilon|/2},$$

where we used that $n_{ij}\le t$, Jensen inequality and that $\sum_{i<j}n_{ij}=|\Upsilon|$ to get the last inequality. The same bound is also true for the absolute value of the contribution of a fixed $t/2$-uple in the second term, up to a combinatorial factor only depending on t. Hence, we write:

$$|R_{k,t}(N)|\le\sum_{\Upsilon}^{*}c(\Upsilon)\Theta(\Upsilon),$$

where now $c(\Upsilon)$ is a combinatorial factor which incorporates the combinatorial factor from before and the number of different terms from the original sums which correspond to Υ (even if we can't say that $c(\Upsilon)$ is independent from N since if N is too small $c(\Upsilon)$ is zero because we only sum over multigraphs with vertices $\{1,...,N\}$, it is still obviously true that $c(\Upsilon) \leq a(t,k)$ uniformly in N). Now we can try to understand over what multigraphs $\sum^*$ runs. For the previous observations we already discarded all the pairwise disjoint double loops, but there are, of course, many multigraphs on $\{1,...,N\}$ which do not correspond to any term in the orginal sums and some which correspond to some terms but whose contribution is actually zero. Indeed, $\sum^*$ is only over multigraphs Υ such that (defining $\hat{n}_i(\Upsilon) = \sum_{j\neq i} n_{ij}(\Upsilon)$):

i) $|\Upsilon| = \sum_{i<j} n_{ij}(\Upsilon) \leq t \cdot k$,

ii) either $n_{ij}(\Upsilon) = 0$ or $2 \leq n_{ij}(\Upsilon) \leq t$,

iii) either $\hat{n}_i(\Upsilon) = 0$ or $\hat{n}_i(\Upsilon) \geq 4$,

iv) there is either a vertex i such that $\hat{n}_i(\Upsilon) \geq 8$ or a couple of vertices i,j such that $\hat{n}_i(\Upsilon), \hat{n}_j(\Upsilon) \geq 6$.

Properties $i)$, $ii)$ and $iii)$ hold since each multigraph is a union of exactly t simple loops of length at most k and by observing that all the multigraphs with at least one edge of multiplicity one have zero contribution by independence and symmetry of the J_{ij}, property $iv)$ is instead a consequence of having already discarded all the multigraphs which consist of pairwise disjoint double loops.

If we fix k and t, up to labeling the vertices, we are summing over uniformly in N finitely many multigraphs. The weight associated to each unnumbered multigraph is independent of the particular labeling and decays with N as $N^{-|\Upsilon|/2}$, whereas the number of labelings of a multigraph grows with N as $N^{|V(\Upsilon)|}$, where $|V(\Upsilon)|$ is the number of vertices in Υ. In particular, by property $iii)$ and $iv)$, we have that:

$$|\Upsilon| = \frac{1}{2}\sum_i \hat{n}_i(\Upsilon) \geq 2|V(\Upsilon)| + 2 \quad \Leftrightarrow \quad |\Upsilon|/2 - |V(\Upsilon)| \geq 1.$$

We stress that for Υ composed of pairwise disjoint double loops property $iv)$ does not hold and thus, as we claimed before, the order in N of its contribution is $O(1)$ since $|\Upsilon|/2 - |V(\Upsilon)| \geq 0$. At the same time, with these computations we also managed to show that the contributions of all the other multigraphs are indeed of order $O\left(\frac{1}{N}\right)$:

$$|R_{k,t}(N)| \leq \sum_{\Upsilon}^{*} c(\Upsilon)\Theta(\Upsilon) \leq \tilde{a}(t,k)N^{-|\Upsilon|/2+|V(\Upsilon)|} \leq \tilde{a}(t,k)N^{-1} \xrightarrow{N\to\infty} 0.$$

This is exactly claim (3.11) and thus the proof of (3.4) and (3.5) is complete. □

To conclude we state the Lemma for convergence in distribution that we used in the previous proof (we refer to [3] for the proof of this Lemma).

Lemma 3.3. *Let X_N be a sequence of random variables such that, for each $\varepsilon > 0$, there exist X_N^ε satisfying:*

(i) $||X_N - X_N^\varepsilon||_2 \leq \varepsilon$,

(ii) $X_N^\varepsilon \xrightarrow{\mathscr{D}} X^\varepsilon$ *for* $N \to \infty$,

then the following limits exist and are equal:

$$\mathscr{D}\text{-}\lim_{N\to\infty} X_N = \mathscr{D}\text{-}\lim_{\varepsilon\to 0} X^\varepsilon.$$

4. THE LOW TEMPERATURE REGIME AND THE GUERRA TONINELLI THEOREM

Again, we will consider an Hamiltonian of the form (2.1), but this time we will work under the assumption that the r.v. J_{ij} are i.i.d standard Gaussians. The following results have actually been generalized to the Bernoulli case by Talagrand in [9] and to the case of any i.i.d J_{ij} with order three moments by Carmona and Yu in [2], but we will stick to the Gaussian case for simplicity. As in the previous section, we are interested in the limit of the free energy $F_N(\beta)$. Here, in the low temperature regime, we will only investigate the existence of the aforementioned limit, which is exactly the content of the following Guerra-Toninelli Theorem.

Theorem 4.1 (Guerra-Toninelli)**.** *For every $\beta > 0$, the limit:*

$$\lim_{N\to\infty} F_N(\beta),$$

exists.

Before explaining the proof, which is pretty short but, as we will see, requires a very good idea, we need some preliminary discussion. We begin by observing that $H(\sigma)$ can be seen as a Gaussian process indexed by $\sigma \in \Sigma_N$ and it is immediate to compute its covariance:

$$\text{(4.1)}\quad \mathbb{E}H_N(\sigma^i)H_N(\sigma^j) = \frac{1}{N}\sum_{1\le k<l\le N} \sigma_k^i\sigma_l^i\sigma_k^j\sigma_l^j = \frac{1}{2N}\left[\left(\sum_{k=1}^N \sigma_k^i\sigma_k^j\right)^2 - \sum_{k=1}^N \left(\sigma_k^i\sigma_k^j\right)^2\right]$$

$$=\frac{1}{2}\left[N\left(\frac{1}{N}\sum_{k=1}^N \sigma_k^i\sigma_k^j\right)^2 - 1\right] = \frac{1}{2}\left(NR_{i,j}^2 - 1\right),$$

which depends only on the renormalized scalar products

$$R_{i,j} := \frac{1}{N}\sum_{k=1}^N \sigma_k^i\sigma_k^j = \frac{1}{N}\langle\sigma^i,\sigma^j\rangle.$$

$R_{i,j}$ is called *overlap* of the two configurations σ^i and σ^j. The fact that the covariance depends only on $R_{i,j}$ is actually very important, as it implies that the Gaussian process is invariant by any orthogonal transformation of the configurations (since orthogonal transformations do not change scalar products). In particular, the so called overlap matrix, with entry ij given by the overlap of σ^i and σ^j, is a crucial object to study when approaching deeper questions about the SK-model, like the structure of the Gibbs measure or the exact value of the limit of the free energy (i.e. the Parisi formula). This was just an interesting remark, since in our dissertation we will only use this formula in one passage of the Guerra-Toninelli theorem, without exploiting all its implications.

Before engaging the theorem we need some tools, namely a technique called Gaussian integration by parts (actually a more or less straightforward consequence of it) and Fekete's Lemma, which is rigorously stated below (we will not prove these results and again we refer to [6] for completeness).

Lemma 4.2 (Gaussian Integration by Parts)**.** *Let $(g_k)_{1\le k\le n}$ be a vector of* jointly *Gaussian random variables and let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous and differentiable function satisfying some mild growth conditions, then:*

$$\mathbb{E}g_1F(g) = \sum_{k\le n}\left(\mathbb{E}g_1g_k\right)\left(\mathbb{E}\frac{\partial F}{\partial x_k}(g)\right).$$

Remark 4.3. We did not specify the growth conditions that F has to satisfy but they are really mild: in particular exponential growth of F and its derivative is enough to guarantee the validity of the previous result.

We will need the following consequence of Gaussian integration by parts.

Corollary 4.4. *Let $(x(\sigma))$ and $(y(\sigma))$ be two jointly Gaussian vectors indexed by $\sigma \in \Sigma$, with Σ finite. Let G be a measure on Σ and let G' be the random Gibbs measure induced by $(y(\sigma))$ on Σ, i.e.:*

$$G'(\sigma) = \frac{\exp y(\sigma)}{\sum_\sigma \exp y(\sigma) G(\sigma)} G(\sigma).$$

If we denote by $\langle \cdot \rangle$ the average with respect to $G'^{\otimes \infty}$ then:

$$\mathbb{E}\langle x(\sigma)\rangle = \mathbb{E}\langle \mathbb{E}x(\sigma^1)y(\sigma^1) - \mathbb{E}x(\sigma^1)y(\sigma^2)\rangle. \tag{4.2}$$

Finally we give the rigorous statement of Fekete's Lemma.

Lemma 4.5 (Fekete's Lemma)**.** *Let x_n be a superadditive sequence (i.e. $x_{n+m} \geq x_n + x_m$ for all n, $m \geq 1$), then:*

$$\lim_{n\to\infty} \frac{x_n}{n} = \sup_{n\geq 1} \frac{x_n}{n}.$$

With these results in hand, we are finally ready to prove the Guerra-Toninelli theorem.

Proof of Theorem 4.1. The strategy of the proof is to show that the sequence $NF_N(\beta)$ is superadditive and then conclude the existence of the limit of $\frac{NF_N(\beta)}{N} = F_N(\beta)$ by Fekete's Lemma 4.5. In order to obtain superadditivity, we fix N and M and perform an interpolation between a system with $N + M$ spins and the product of two independent systems with N and M spins, respectively. Precisely, by "independent systems" we mean that the three associated Hamiltonians of the form (2.1): $H_N(\rho)$, $H_M(\tau)$ and $H_{N+M}(\sigma)$, respectively defined on Σ_N, Σ_M and $\Sigma_{N+M} = \Sigma_N \times \Sigma_M$, are independent. Of course, we can represent $\sigma \in \Sigma_{N+M}$ as $\sigma = (\rho, \tau)$ with $\rho \in \Sigma_N$ and $\tau \in \Sigma_M$. For every $t \in [0, 1]$, we define the following Hamiltonian on Σ_{N+M}:

$$H_t(\sigma) := \sqrt{t} H_{N+M}(\sigma) + \sqrt{1-t}\left(H_N(\rho) + H_M(\tau)\right).$$

We observe that H_t is just interpolating between the full interacting system on $N+M$ spins and the system where the first N are decoupled with the second M spins. Consequently, we expect the corresponding free energy to be an interpolation between the free energy on $N + M$ spins and the sum of the free energies on N and M spins. Indeed, this is true up to constant factors: if we denote by Z_t the partition function corresponding to H_t (we are dropping the dependence on β in our notation as it is irrelevant for all the following computations) and if we look at the associated free energy, i.e.:

$$\phi(t) = \frac{1}{N+M} \mathbb{E} \log Z_t,$$

it is then immediate to check that:

$$\phi(0) = \frac{N}{N+M} F_N(\beta) + \frac{M}{N+M} F_M(\beta) \quad \text{and} \quad \phi(1) = F_{N+M}.$$

From this, it follows that superadditivity of F_N is equivalent to $\phi(0) \leq \phi(1)$. Hence, the introduction of H_t has given us a very practical way to check the superadditivity of F_N, our initial goal. We will indeed be able to show $\phi(0) \leq \phi(1)$ and actually something more: we will show that $\phi'(t) \geq 0$ for all $t \in [0,1]$. We define G_t to be the random Gibbs measure associated to H_t and denote by $\langle \cdot \rangle_t$ the average with respect to the measure $G_t^{\bigotimes \infty}$, we then proceed to compute $\phi'(t)$:

$$\phi'(t) = \frac{1}{N+M}\mathbb{E}\left[\frac{1}{Z_t}\sum_{\sigma \in \Sigma_{N+M}} \frac{\partial H_t(\sigma)}{\partial t} \exp H_t(\sigma)\right] = \frac{1}{N+M}\mathbb{E}\left\langle \frac{\partial H_t(\sigma)}{\partial t}\right\rangle_t \tag{4.3}$$

$$= \frac{1}{N+M}\mathbb{E}\left\langle \mathbb{E}\frac{\partial H_t(\sigma_1)}{\partial t} H_t(\sigma_1) - \mathbb{E}\frac{\partial H_t(\sigma_1)}{\partial t} H_t(\sigma_2)\right\rangle_t .$$

Here in the last passage we used Gaussian integration by parts, in the form (4.2).

Given $\sigma^1 = (\rho^1, \tau^1)$ and $\sigma^2 = (\rho^2, \tau^2)$, we denote the overlaps of ρ^1, ρ^2 and of τ^1, τ^2 by:

$$R^1_{1,2} = \frac{1}{N}\sum_{i=1}^N \rho_i^1 \rho_i^2 \quad \text{and} \quad R^2_{1,2} = \frac{1}{M}\sum_{i=1}^N \tau_i^1 \tau_i^2.$$

Using the independece of H_{N+M}, H_N and H_M and formula (4.1) for the covariance we get, observing that $R_{1,1} = R^1_{1,1} = R^2_{1,1} = 1$:

$$\mathbb{E}\frac{\partial H_t(\sigma_1)}{\partial t} H_t(\sigma_1) = \frac{N+M}{4}\left(R_{1,1}\right)^2 + \frac{1}{4} - \frac{N}{4}\left(R^1_{1,1}\right)^2 - \frac{M}{4}\left(R^2_{1,1}\right)^2 = \frac{1}{4}, \tag{4.4}$$

$$\mathbb{E}\frac{\partial H_t(\sigma_1)}{\partial t} H_t(\sigma_2) = \frac{N+M}{4}\left(R_{1,2}\right)^2 + \frac{1}{4} - \frac{N}{4}\left(R^1_{1,2}\right)^2 - \frac{M}{4}\left(R^2_{1,2}\right)^2 . \tag{4.5}$$

Thus, applying (4.4) and (4.5) to (4.3), we have:

$$\phi'(t) = \frac{1}{N+M}\mathbb{E}\left\langle \frac{1}{4} - \frac{N+M}{4}\left(R_{1,2}\right)^2 - \frac{1}{4} + \frac{N}{4}\left(R^1_{1,2}\right)^2 + \frac{M}{4}\left(R^2_{1,2}\right)^2\right\rangle_t$$

$$= -\frac{1}{4}\mathbb{E}\left\langle \left(R_{1,2}\right)^2 - \frac{N}{N+M}(R^1_{1,2})^2 - \frac{M}{N+M}(R^2_{1,2})^2\right\rangle_t .$$

To conclude, we observe that $R_{1,2} = \frac{N}{N+M}R^1_{1,2} + \frac{M}{N+M}R^2_{1,2}$ and thus, by convexity of $r \mapsto r^2$:

$$\phi'(t) \geq 0.$$

Hence, as claimed, we have $\phi(0) \leq \phi(1)$ or, equivalently, the superadditivity of NF_N and we can complete the proof by applying Fekete's Lemma. □

REFERENCES

[1] M. AIZENMAN, J. L. LEBOWITZ, AND D. RUELLE, *Some rigorous results on the Sherrington-Kirkpatrick spin glass model*, Communications in mathematical physics, 112 (1987), pp. 3–20.

[2] P. CARMONA AND Y. HU, *Universality in Sherrington–Kirkpatrick's spin glass model*, in Annales de l'Institut Henri Poincare (B) Probability and Statistics, vol. 42, Elsevier, 2006, pp. 215–222.

[3] W. FELLER, *An introduction to probability theory and its applications*, vol. 2, John Wiley & Sons, 2008.

[4] F. GUERRA AND F. L. TONINELLI, *The thermodynamic limit in mean field spin glass models*, Communications in Mathematical Physics, 230 (2002), pp. 71–79.

[5] S. KIRKPATRICK AND D. SHERRINGTON, *Solvable model of a spin-glass*, Phys. Rev. Lett, 35 (1975), pp. 1792–1796.

[6] D. PANCHENKO, *The Sherrington-Kirkpatrick model*, Springer Science & Business Media, 2013.

[7] G. PARISI, *Infinite number of order parameters for spin-glasses*, Physical Review Letters, 43 (1979), p. 1754.
[8] ———, *A sequence of approximated solutions to the SK model for spin glasses*, Journal of Physics A: Mathematical and General, 13 (1980), p. L115.
[9] M. TALAGRAND, *Gaussian averages, Bernoulli averages, and Gibbs' measures*, Random Structures & Algorithms, 21 (2002), pp. 197–204.
[10] ———, *The Parisi formula*, Annals of mathematics, (2006), pp. 221–263.