On the operator norm of a Hermitian random matrix with correlated entries

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Abstract

We consider a correlated $N \times N$ Hermitian random matrix with a polynomially decaying metric correlation structure. By calculating the trace of the moments of the matrix and using the summable decay of the cumulants, we show that its operator norm is stochastically dominated by one.

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1 Introduction

Let H be a Hermitian $N \times N$ random matrix such that $H = \frac{1}{\sqrt{N}}W$, where $W \in \mathbb{C}^{N\times N}$ has matrix elements of order one. For Wigner matrices, i.e., when the entries of W are identically distributed and independent (up to the Hermitian symmetry) with some mild moment condition, it is well known that $\|H\|$ is bounded uniformly in N with very high probability. In fact, it even converges to 2 under the normalization $\mathbb{E}|W_{ij}|^2 = 1$ (see [2] and [1, Thm 2.1.22], as well as [5], [6], [7] for more quantitative bounds under stronger moment conditions). In contrast, if the entries of W are very strongly correlated, the norm of H may be as large as $\sqrt{N}$. In this paper, we assume the entries of W to be correlated following a *polynomially decaying metric correlation structure* as, e.g., considered in [4], but with a weaker, summable correlation decay. This dependence structure is characterized by the 2-cumulants of the matrix elements W_{ij} decaying at least as an inverse $2+\varepsilon$ power of the distance with respect to a natural metric on the index pairs (i,j). The higher cumulants follow a similar pattern (see Assumption (A3) below). Under these mild decay conditions, we show that $\|H\|$ is essentially bounded with very high probability.

This result was already stated in [4], indicating that an extension of Wigner's moment method applies. In the current paper, we carry out this task which turns out to be rather involved. In [4], this bound was used as an apriori control on $\|H\|$ for the resolvent method, leading to optimal local laws for H. We remark that it is possible the modify the proof in [4] to obtain the bound on $\|H\|$ directly, i.e., without relying on the current paper. However, an independent proof via the moment method has several advantages. First, it is conceptually much simpler and less technical than the resolvent approach in [4]. Further, it only requires the summability of the 2-cumulants (see exponent $s > 2$ in (1) below), while [4] assumed a faster decay ($s > 12$, see [4, Eq. (3a)]). Lastly, the current method can be generalized to even weaker correlation decays, resulting in correlated random matrix ensembles whose norm grows with N but slower than the trivial $\sqrt{N}$ bound.

We start by giving some general notation and the precise assumptions on the matrix W in Section 1.1 below. The bound on the operator norm of H is then formulated in Theorem 1 and its proof is given in Sections 2 and 3. For simplicity, the argument is carried out only for symmetric $H \in \mathbb{R}^{N\times N}$. The Hermitian case follows analogously and is hence omitted.

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1.1 Notation and Assumptions on the Model

Throughout the paper, boldface indicates vectors $\mathbf{x} \in \mathbb{C}^N$ and their Euclidean norm is denoted by $\|\mathbf{x}\|_2$. Further, the operator norm of a matrix $A \in \mathbb{C}^{N\times N}$ is denoted by $\|A\|$. In the estimates, C (without subscript) denotes a generic constant the value of which may change from line to line. We note the following assumptions on the matrix W.

Assumption (A1) $\mathbb{E}W_{i,j} = 0$ *for all* $i, j = 1, \dots, N$.

Assumption (A2) *For all* $q \in \mathbb{N}$ *there exists a constant* μ_q *such that* $\mathbb{E}|W_{i,j}|^q \le \mu_q$ *for all* $i, j = 1, \dots, N$.

The assumption on the correlation decay is given in terms of the multivariate cumulants $\kappa^{(k)}$ of the matrix elements.

Definition (Cumulants)**.** *Let* $\mathbf{w} = (w_1, \dots, w_n)$ *be a random vector taking values in* $\mathbb{R}^n$. *The* ***cumulants*** κ_m *of* $\mathbf{w}$ *are defined as the Taylor coefficients of the log-characteristic function of* $\mathbf{w}$, *i.e.,*

$$\ln \mathbb{E}[\mathrm{e}^{i\mathbf{t}\cdot\mathbf{w}}] = \sum_m \kappa_m \frac{(i\mathbf{t})^m}{m!},$$

where the sum is taken over all multi-indices $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ *and* $m! = \prod_{j=1}^n (m_j!)$. *For a multiset* $B \subset \{1, \dots, n\}$ *with* $|B| = k$, *we also write* $\kappa^{(k)}(w_j | j \in B)$ *instead of* κ_m, *where* m_i *is the multiplicity of* $m_i \in B$.

The complex cumulants arising whenever $\mathbf{w}$ takes values in $\mathbb{C}^n$ are defined by considering the real and imaginary part of the random variables separately. To keep the notation short, we usually view the cumulants as a function of the indices of the matrix elements by identifying $\kappa^{(k)}(W_{a_1,a_2}, \dots)$ with $\kappa^{(k)}(a_1a_2, \dots)$ or $\kappa^{(k)}(\alpha_1, \dots, \alpha_k)$ using $\alpha_1, \dots, \alpha_k \in \{1, \dots, N\}^2$. Further, d denotes the Euclidean distance on $\{1, \dots, N\}^2$ modulo the (Hermitian) symmetry, i.e.,

$$d(a_1a_2, a_3a_4) := \min\{|a_1 - a_3| + |a_2 - a_4|, |a_1 - a_4| + |a_2 - a_3|\}.$$

In this notation, the conditions from the polynomially decaying metric correlation structure can be formulated as follows.

Assumption (A3) *The* k*-cumulants* $\kappa^{(k)}$ *of the matrix elements of* W *satisfy*

$$|\kappa^{(2)}(a_1a_2, a_3a_4)| \le \frac{C_\kappa}{1 + d(a_1a_2, a_3a_4)^s}, \tag{1}$$

$$|\kappa^{(k)}(\alpha_1, \dots, \alpha_k)| \le C(k) \prod_{e\in T_{min}} |\kappa^{(2)}(e)|, \ k \ge 3, \tag{2}$$

for $s > 2$ *and some constants* $C_\kappa, C(k) > 0$. *Here,* T_{min} *is the minimal spanning tree on the complete graph on* k *vertices labelled by* $\alpha_1, \dots, \alpha_k$ *with edge weights* $d(\alpha_i, \alpha_j)$, *i.e., the spanning tree for which the sum of the edge weights is minimal.*

Note that a correlation decay of the form (2) arises in different statistical physics models (see [3]).

1.2 Statement of the Main Result

With the notation established, we give the statement on the operator norm of H as follows.

Theorem 1. *Under the assumptions* (A1)-(A3)*, we have that for all* $\varepsilon > 0$*,* $D > 0$ *there exists a suitable constant* $C(\varepsilon, D)$ *such that, for all* $N \in \mathbb{N}$*,*

$$\mathbb{P}\big(\|H\| > N^{\varepsilon}\big) \leq C(\varepsilon, D)N^{-D}.$$

1.3 Setup for the Moment Method

We show that the assumptions (A1)-(A3) imply that

$$\mathbb{E}[\tfrac{1}{N}\mathrm{tr}(H^k)] \leq C(k), \quad \forall k \in \mathbb{N}, \tag{3}$$

from which the statement of Theorem 1 follows by an application of Chebyshev's inequality. As H is Hermitian, we have $\|H\|^k \leq \mathrm{tr}(H^k)$ for all even $k \in \mathbb{N}$. This implies

$$\mathbb{P}(\|H\| > N^{\varepsilon}) \leq \frac{\mathbb{E}[\|H\|^k]}{N^{k\varepsilon}} \leq \frac{N\mathbb{E}[\frac{1}{N}\mathrm{tr}(H^k)]}{N^{k\varepsilon}} \leq \frac{NC(k)}{N^{k\varepsilon}}$$

and thus gives the desired bound if k is chosen large enough.

Writing out the term on the left-hand side of (3) using a cumulant expansion yields

$$\begin{aligned}\mathbb{E}[\tfrac{1}{N}\mathrm{tr}(H^k)] &= \frac{1}{N}\sum_{a_1,\ldots,a_k} \mathbb{E}[H_{a_1,a_2}H_{a_2,a_3}\ldots H_{a_k,a_1}] \\ &= N^{-k/2-1}\sum_{\pi\in\Pi_k}\sum_{a_1,\ldots,a_k}\prod_{B\in\pi}\kappa^{(|B|)}(a_ja_{j+1}|j\in B),\end{aligned} \tag{4}$$

where Π_k denotes the set of partitions of $\{1,\ldots,k\}$ and the index $j+1$ is to be interpreted mod k, i.e., if $j = k$, then $a_ja_{j+1} = a_ka_1$. Observe that all terms involving 1-cumulants vanish due to (A1). Hence, one can restrict the sum to partitions π without singleton sets. The cumulant expansion (4) is the main difference between the real symmetric and complex Hermitian case, as considering $H \in \mathbb{C}^{N\times N}$ requires replacing the cumulants by their complex counterparts. However, one can always reduce the argument to the real case by considering the real and imaginary parts of the random variables separately. Thus, from now on we assume H to be real. We develop the estimates by first deriving suitable bounds for the products that only involve 2-cumulants, which include the leading terms, and then successively incorporating higher-order cumulants. The fast correlation decay from Assumption (A3) implies that roughly half of the summations over the indices $a_1,\ldots,a_k$ yield a factor N, while the other half can be summed up with an N-independent bound. Therefore, we introduce the following counting rule.

Counting Rule (CR). Every independent summation over an index $a_1,\ldots,a_k$ in (4) yields a contribution of order $\sqrt{N}$.

Note that bounding the leading terms requires an extra power of N (see the proof of (5) below) such that the factor $N^{-k/2-1}$ in (4) is canceled out completely.

2 Proof of (3) for Terms Involving Only 2-Cumulants

Throughout this section, assume that k is even and $|B| = 2$ for all $B \in \pi$. We aim to show that, for all such $\pi \in \Pi_k$,

$$N^{-k/2-1}\Big|\sum_{a_1,\ldots,a_k}\prod_{B\in\pi}\kappa^{(2)}(a_ja_{j+1}|j\in B)\Big| \le C(k). \tag{5}$$

The terms can be visualized by considering a k-gon, where the vertices are labeled by the indices $a_1, \ldots, a_k$ and the edges by the successive double indices $(a_1, a_2), \ldots, (a_k, a_1)$. We denote the corresponding graph by Γ_k. In this picture, every 2-cumulant combines two edges such that each edge belongs to exactly one 2-cumulant. We say that a vertex *belongs to a 2-cumulant*, if it is adjacent to one of the edges associated with it. A vertex (resp. the corresponding index) which only occurs in a single 2-cumulant is referred to as *internal vertex* (resp. *internal index*). We give an example for a particular $\pi \in \Pi_8$ in Fig. 1 below, where different linestyles indicate the edges associated with the same 2-cumulant. Further examples that include internal indices are given in Fig. 2.

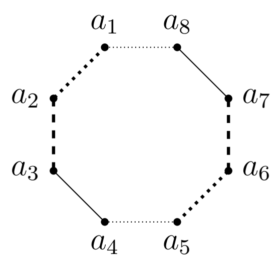

Fig. 1. Visualization of $\kappa^{(2)}(a_1a_2, a_5a_6)\kappa^{(2)}(a_2a_3, a_6a_7)\kappa^{(2)}(a_3a_4, a_7a_8)\kappa^{(2)}(a_4a_5, a_8a_1)$

Assume first that π is chosen such that the 2-cumulants occurring in the term do not involve internal indices. We note the following general estimates whose proofs are elementary from (1) and the fact that $s > 2$.

Lemma 2. *Assume that* (A3) *holds. Then*

$$\sum_{a_1}|\kappa^{(2)}(a_1a_2, a_3a_4)| \le C, \quad \sum_{a_1,a_2}|\kappa^{(2)}(a_1a_2, a_3a_4)| \le C, \quad \sum_{a_1,a_3}|\kappa^{(2)}(a_1a_2, a_3a_4)| \le CN,$$
$$\sum_{a_1,a_2,a_3}|\kappa^{(2)}(a_1a_2, a_3a_4)| \le CN, \quad \sum_{a_1,\ldots,a_4}|\kappa^{(2)}(a_1a_2, a_3a_4)| \le CN^2$$

uniformly for any choice of the unsummed indices. In particular, the estimates follow (CR).

Note that summation over an internal index would, in general, not obey this counting rule, since $\sum_{a_2=1}^{N}|\kappa^{(2)}(a_1a_2, a_2a_3)|$ may be of order N if $a_1 = a_3$. This is the reason why internal indices are treated separately. The key to estimating (5) in the given case is a recursive summation procedure. We demonstrate the approach for the term visualized in Fig. 1.

Example 3. To start the summation, estimate

$$\begin{aligned} S &:= N^{-5}\sum_{a_1,\ldots,a_8}|\kappa^{(2)}(a_1a_2, a_5a_6)\kappa^{(2)}(a_2a_3, a_6a_7)\kappa^{(2)}(a_3a_4, a_7a_8)\kappa^{(2)}(a_4a_5, a_8a_1)| \\ &\le N^{-5}\sum_{a_1,\ldots,a_8,a_1'}|\kappa^{(2)}(a_1a_2, a_5a_6)\kappa^{(2)}(a_2a_3, a_6a_7)\kappa^{(2)}(a_3a_4, a_7a_8)\kappa^{(2)}(a_4a_5, a_8a_1')|. \end{aligned} \tag{6}$$

Adding the extra summation label a_1' appears as an unnecessary overestimate, but it simplifies the following steps by breaking the cyclic structure of the graph. Next, isolate the 2-cumulant involving a_1 by taking the maximum over the remaining indices a_2, a_4, a_5 for the other factors. Together with a_1, we sum over all labels appearing in $\kappa^{(2)}(a_1a_2, a_5a_6)$. Applying the last bound in Lemma 2 yields a factor $(\sqrt{N})^4 = N^2$ and the estimate

$$S \leq N^{-5}CN^2 \max_{a_2,a_5,a_6} \Big(\sum_{a_3,a_4,a_7,a_8,a_1'} |\kappa^{(2)}(a_2a_3, a_6a_7)\kappa^{(2)}(a_3a_4, a_7a_8)\kappa^{(2)}(a_4a_5, a_8a_1')| \Big).$$

From a_1, continue counter-clockwise along the octagon to find the next index to sum over, i.e., a_3. Identifying and isolating $\kappa^{(2)}(a_2a_3, a_6a_7)$, sum over all remaining indices in the factor, i.e., a_3, a_6 and a_7, to obtain a contribution of $(\sqrt{N})^3 = N^{3/2}$ and the estimate

$$S \leq CN^{-3/2} \max_{a_3,a_4,a_5,a_7} \Big(\sum_{a_8,a_1'} |\kappa^{(2)}(a_3a_4, a_7a_8)\kappa^{(2)}(a_4a_5, a_8a_1')| \Big).$$

Repeating the previous step, continuing along the octagon yields a_8 as the next index. Isolating $\kappa^{(2)}(a_4a_5, a_8a_1')$ and performing the last two summations yields a factor of $(\sqrt{N})^2 = N$ and the final estimate

$$S \leq CN^{-1/2} \max_{a_3,a_4,a_7,a_8} |\kappa^{(2)}(a_3a_4, a_7a_8)|,$$

which can be bounded by a constant as claimed in (5). Note that the pairing in Fig. 1 gives a subleading contribution to (4).

As this recursive summation procedure relies on Lemma 2, it cannot be applied directly if summation over internal indices are present. To prepare for the general case, we first extend Lemma 2 to estimates for 2-cumulants where the unsummed indices are replaced by arbitrary fixed vectors $\mathbf{x} \in \mathbb{R}^N$ in the sense that $\kappa^{(2)}(\mathbf{x}a_2, a_3a_4) := \sum_{a_1=1}^N \kappa^{(2)}(a_1a_2, a_3a_4)x_{a_1}$, and terms such as $\kappa^{(2)}(\mathbf{x}a_2, a_3\mathbf{y})$ are defined similarly. We collect some of the estimates below, where we follow the convention that vectors only occur in place of the first index of an index pair. However, the same bounds hold if the second index of the respective pair is replaced instead. As in Lemma 2, no internal index is summed up.

Lemma 4. *Assume that* (A3) *holds and let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. *Then we have*

$$\sum_{a_2} |\kappa^{(2)}(\mathbf{x}a_2, a_3a_4)| \leq C\|\mathbf{x}\|_2, \quad \sum_{a_3} |\kappa^{(2)}(\mathbf{x}a_2, a_3a_4)| \leq CN^{1/2}\|\mathbf{x}\|_2,$$
$$\sum_{a_2,a_3} |\kappa^{(2)}(\mathbf{x}a_2, a_3a_4)| \leq CN\|\mathbf{x}\|_2, \quad \sum_{a_3,a_4} |\kappa^{(2)}(\mathbf{x}a_2, a_3a_4)| \leq CN\|\mathbf{x}\|_2,$$
$$\sum_{a_2,a_3,a_4} |\kappa^{(2)}(\mathbf{x}a_2, a_3a_4)| \leq CN^{3/2}\|\mathbf{x}\|_2$$

uniformly for any choice of unsummed indices. Similar bounds hold with two vectors, i.e.,

$$|\kappa^{(2)}(\mathbf{x}a_2, \mathbf{y}a_4)| \leq C\|\mathbf{x}\|_2\|\mathbf{y}\|_2, \quad \sum_{a_2} |\kappa^{(2)}(\mathbf{x}a_2, \mathbf{y}a_4)| \leq CN^{1/2}\|\mathbf{x}\|_2\|\mathbf{y}\|_2,$$
$$\sum_{a_2,a_4} |\kappa^{(2)}(\mathbf{x}a_2, \mathbf{y}a_4)| \leq CN\|\mathbf{x}\|_2\|\mathbf{y}\|_2.$$

In particular, the estimates follow (CR).

Proof. Noting that $\max_j |x_j| \le \|\mathbf{x}\|_2$, the bounds in Lemma 2 imply that

$$\sum_{a_2} |\kappa^{(2)}(\mathbf{x}a_2, a_3a_4)| \le \|\mathbf{x}\|_2 \sum_{a_1,a_2} |\kappa^{(2)}(a_1a_2, a_3a_4)| \le C\|\mathbf{x}\|_2.$$

If the summation over a_2 is replaced by a summation over a_3 or a_4, we obtain a bound of order N instead. Further, an application of the Cauchy-Schwarz inequality leads to

$$\sum_{a_3} |\kappa^{(2)}(\mathbf{x}a_2, a_3a_4)| \le \|\mathbf{x}\|_2 \sqrt{\sum_{a_1} \Big(\sum_{a_3} |\kappa^{(2)}(a_1a_2, a_3a_4)| \Big)^2} \le CN^{1/2}\|\mathbf{x}\|_2. \tag{7}$$

Recalling that the summation over three indices gives a factor N, the bound for summation over all three indices in $|\kappa^{(2)}(\mathbf{x}a_2, a_3a_4)|$ follows analogously.

For 2-cumulants that involve two vectors, applying (1) yields

$$|\kappa^{(2)}(\mathbf{x}a_2, \mathbf{y}a_4)| \le \sum_{a_1,a_3} \Big(\frac{C_\kappa |x_{a_1} y_{a_3}|}{1 + |a_1 - a_3|^s + |a_2 - a_4|^s} + \frac{C_\kappa |x_{a_1} y_{a_3}|}{1 + |a_1 - a_4|^s + |a_2 - a_3|^s} \Big). \tag{8}$$

Set $\varepsilon = \|\mathbf{y}\|_2 / \|\mathbf{x}\|_2$ and estimate the first term as

$$\sum_{a_1,a_3} \frac{|x_{a_1} y_{a_3}|}{1 + |a_1 - a_3|^s + |a_2 - a_4|^s} \le \sum_{a_1,a_3} \frac{\varepsilon |x_{a_1}|^2 + \varepsilon^{-1} |y_{a_3}|^2}{1 + |a_1 - a_3|^s} \le C(\varepsilon \|\mathbf{x}\|_2^2 + \varepsilon^{-1} \|\mathbf{y}\|_2^2)$$

to obtain an N-independent bound. The estimate of the second term is similar.

Adding a summation over one index, e.g., a_2, two applications of the Cauchy-Schwarz inequality and the third estimate of Lemma 2 yield

$$\sum_{a_2} |\kappa^{(2)}(\mathbf{x}a_2, \mathbf{y}a_4)| \le \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \sqrt{\sum_{a_3} \Big(\sum_{a_1} \sum_{a_2} |\kappa^{(2)}(a_1a_2, a_3a_4)| \Big)^2} \le CN^{1/2} \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

Finally, it follows

$$\sum_{a_2,a_4} |\kappa^{(2)}(\mathbf{x}a_2, \mathbf{y}a_4)| \le \sum_{a_1,\ldots,a_4} \Big(\frac{C_\kappa |x_{a_1} y_{a_3}|}{1 + |a_1 - a_3|^s + |a_2 - a_4|^s} + \frac{C_\kappa |x_{a_1} y_{a_3}|}{1 + |a_1 - a_4|^s + |a_2 - a_3|^s} \Big).$$

Here, we obtain

$$\sum_{a_1,\ldots,a_4=1}^{N} \frac{|x_{a_1} y_{a_3}|}{1 + |a_1 - a_3|^s + |a_2 - a_4|^s} \le CN \sum_{a_1,a_3=1}^{N} \frac{|x_{a_1} y_{a_3}|}{1 + |a_1 - a_3|^s} \le CN \|\mathbf{x}\|_2 \|\mathbf{y}\|_2,$$

and the estimate for the second term is similar. □

Lemma 4 is the key tool for estimating terms that include summation over internal indices. Consider the matrix $T \in \mathbb{R}^{N \times N}$ defined by its matrix elements

$$T_{a_1,a_3} := \sum_{a_2} T^{(a_2)}_{a_1,a_3} := \sum_{a_2} \kappa^{(2)}(a_1a_2, a_2a_3) \tag{9}$$

and observe that $|T_{a_1,a_3}| \le C_\kappa N$ by (1), but also $\|T\| \le CN$, since

$$\|T^{(a_2)}\mathbf{x}\|_2^2 \le \sum_{a_1} \Big(\sum_{a_3} |\kappa^{(2)}(a_1a_2, a_2a_3) x_{a_3}| \Big)^2 \le \Big(\sum_{a_1} \sum_{a_3} |\kappa^{(2)}(a_1a_2, a_2a_3)|^2 \Big) \|\mathbf{x}\|_2^2 \le C\|\mathbf{x}\|_2^2$$

for $\mathbf{x} \in \mathbb{R}^N$, and $\|T\| \leq \sum_{a_2} \|T^{(a_2)}\|$. Next, we derive similar estimates for the matrix $T^{[j]} \in \mathbb{R}^{N\times N}$ defined for $2 \leq j \leq k-1$ by

$$T^{[j]}_{a_1,a_{2j+1}} := \sum_{a_2,a_{2j}} \kappa^{(2)}(a_1a_2, a_{2j}a_{2j+1}) T^{j-1}_{a_2,a_{2j}}. \tag{10}$$

Note that the superscript corresponds to the total number of 2-cumulants that are rewritten to obtain $T^{[j]}_{a_1,a_{2j+1}}$. Again, we have $|T^{[j]}_{a_1,a_{2j+1}}| \leq CN^j$ by a direct estimate, but also

$$\|T^{[j]}\| \leq CN^j \quad \forall j \in \{2,\dots,k-1\}. \tag{11}$$

As the argument is the same in the general case, consider only $j=2$. Define the vector $\mathbf{y}^{a_4}$ through $(\mathbf{y}^{a_4})_{a_2} := N^{-1}T_{a_2,a_4}$, $a_2 = 1,\dots,N$. Observing that $\|\mathbf{y}^{a_4}\|_2 \leq C$ uniformly in a_4, it follows for $T^{[2],a_4} := \sum_{a_2} \kappa^{(2)}(a_1a_2, a_4a_5)T_{a_2,a_4}$ that

$$\|T^{[2],a_4}\mathbf{x}\|_2^2 \leq \Big(N|\kappa^{(2)}(a_1\mathbf{y}^{a_4}, a_4\mathbf{x})|\Big)^2 \leq CN^2\|\mathbf{y}^{a_4}\|_2^2\|\mathbf{x}\|_2^2 \leq CN^2\|\mathbf{x}\|_2^2 \tag{12}$$

for $\mathbf{x} \in \mathbb{R}^N$, which implies $\|T^{[2]}\| \leq N\max_{a_4}\|T^{[2],a_4}\| \leq CN^2$. Note that the structure of the matrix-vector multiplication in (12) does not allow for the usual convention of the vector occurring only as the first index of an index pair.

We demonstrate the approach for treating general products of 2-cumulants for the terms visualized in Fig. 2 below.

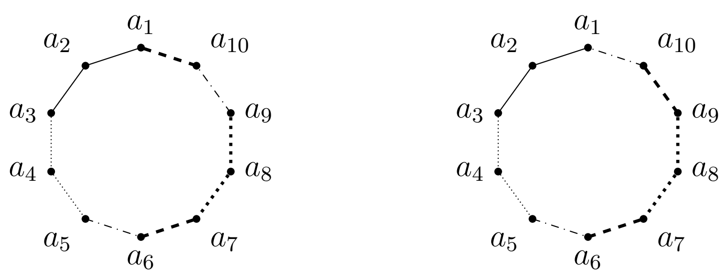

Fig. 2. A crossing (left) and a non-crossing pairing (right) for $k = 10$.

Example 5. First, consider the term on the left of Fig. 2. Recalling the definition of the matrix T from (9), rewrite the summation over the internal indices a_2, a_4 and a_7 as

$$\begin{aligned}
&N^{-6}\sum_{a_1,\dots,a_{10}} \kappa^{(2)}(a_1a_2,a_2a_3)\kappa^{(2)}(a_3a_4,a_4a_5)\kappa^{(2)}(a_5a_6,a_9a_{10})\kappa^{(2)}(a_6a_7,a_{10}a_1)\kappa^{(2)}(a_7a_8,a_8a_9)\\
&= N^{-6}\sum_{a_1,a_5,a_6,a_7,a_9,a_{10}} T^2_{a_1,a_5}\kappa^{(2)}(a_5a_6,a_9a_{10})\kappa^{(2)}(a_6a_7,a_{10}a_1)T_{a_7,a_9}\\
&= N^{-3}\sum_{a_1,a_6,a_7,a_{10}} \kappa^{(2)}(\mathbf{x}^{a_1}a_6,\mathbf{y}^{a_7}a_{10})\kappa^{(2)}(a_6a_7,a_{10}a_1),
\end{aligned} \tag{13}$$

where the vectors $\mathbf{x}^{a_1}, \mathbf{y}^{a_7} \in \mathbb{R}^N$ in the last step are given by

$$x^{a_1}_{a_5} = \frac{1}{N^2}T^2_{a_1,a_5},\ a_5 = 1,\dots,N, \quad y^{a_7}_{a_9} = \frac{1}{N}T_{a_7,a_9},\ a_9 = 1,\dots,N.$$

To keep the notation consistent with the proof in the general case and Lemma 4, we introduce the convention that vectors obtained from matrix elements are always defined via the rows of the respective matrix. Recalling that $\|T\| \leq CN$, we have $\|\mathbf{x}^{a_1}\|_2, \|\mathbf{y}^{a_7}\|_2 \leq C$ uniformly for any choice of a_1 and a_7, respectively. Note that one factor of N per power of T is written in front of the sum and that the convention chosen for the vectors ensures that we always replace the first index of an index pair. Further, the sum obtained from (13) does not involve summation over internal indices. Modifying the recursive summation procedure from Example 3 by also taking the maximum over a_1 in the 2-cumulant involving $\mathbf{x}^{a_1}$ instead of introducing the additional summation label a_1', and using the bounds from Lemmas 2 and 4 yields

$$\begin{aligned}
&N^{-3} \sum_{a_1,a_6,a_7,a_{10}} |\kappa^{(2)}(\mathbf{x}^{a_1} a_6, \mathbf{y}^{a_7} a_{10}) \kappa^{(2)}(a_6 a_7, a_{10} a_1)| \\
&\leq N^{-3} \max_{a_1,a_6,a_7,a_{10}} |\kappa^{(2)}(\mathbf{x}^{a_1} a_6, \mathbf{y}^{a_7} a_{10})| \sum_{a_1,a_6,a_7,a_{10}} |\kappa^{(2)}(a_6 a_7, a_{10} a_1)| \\
&\leq CN^{-3} N^2 \max_{a_1,a_6,a_7,a_{10}} |\kappa^{(2)}(\mathbf{x}^{a_1} a_6, \mathbf{y}^{a_7} a_{10})| \leq CN^{-1} \leq C,
\end{aligned} \tag{14}$$

showing that the pairing on the left of Fig. 2 gives a sub-leading contribution to (4).

In contrast, observe that the non-crossing pairing on the right of Fig. 2 needs to be handled differently, since treating T as before yields $\kappa^{(2)}(\mathbf{x}^{a_1} a_6, a_{10} a_1)$ and $\kappa^{(2)}(a_6 a_7, \mathbf{y}^{a_7} a_{10})$, which cannot be estimated using Lemma 4 due to the indices a_1 and a_7 appearing twice in the respective 2-cumulants. However, the terms can be rewritten using the matrices $T^{[j]}$ defined in (10). Recalling that $\|T^{[j]}\| \leq CN^j$ from (11), we obtain

$$\begin{aligned}
&N^{-6} \Big| \sum_{a_1,a_5,a_6,a_7,a_9,a_{10}} T^2_{a_1,a_5} \kappa^{(2)}(a_5 a_6, a_{10} a_1) \kappa^{(2)}(a_6 a_7, a_9 a_{10}) T_{a_7,a_9} \Big| \\
&= N^{-6} \Big| \sum_{a_6,a_{10}} T^{[3]}_{a_{10},a_6} T^{[2]}_{a_6,a_{10}} \Big| = N^{-6} \big| \mathrm{tr}\big(T^{[3]} T^{[2]}\big) \big| \leq CN^{-6} N N^3 N^2 = C.
\end{aligned}$$

Hence, the term on the left of Fig. 2 yields a leading contribution to (4).

After all these preparations, we can give a complete proof of (5).

2.1 Proof of (5)

Let $k \geq 2$ be even and $\pi \in \Pi_k$ such that $|B| = 2$ for all $B \in \pi$. Recalling that the product on the left-hand side of (5) can be visualized on the graph Γ_k by marking the two edges belonging to the same 2-cumulant in the same color, consider the (ordered) set of edges $E_k = \{(a_1, a_2), \ldots, (a_k, a_1)\}$ and the mapping $\varphi : E_k \to E_k$ that maps every edge to the other one associated with the same 2-cumulant. Starting with $e_1 = (a_1, a_2)$, go through the elements of E_k and note the pairing $(e_n, \varphi(e_n))$ whenever $e_n \neq \varphi(e_i)$ for all $i < n$ to obtain $k/2$ pairs of edges $(e_n, \varphi(e_n))$. We denote these pairings as an (ordered) set $C(\pi) := \{(e_1, \varphi(e_1)), \ldots, (e_{k/2}, \varphi(e_{k/2}))\}$ and refer to the edges in $C(\pi)$ as paired edges. Note that, by construction, every element of E_k occurs exactly once in $C(\pi)$. The graph in which the pairings of $\pi \in \Pi_k$ are marked is denoted by the tuple $(\Gamma_k, C(\pi))$.

The proof of (5) is structured into two main steps. In the first step, we generalize the strategies from Example 5 to rewrite the term on the left-hand side to a form that is tractable by a recursive summation procedure. We then carry out the required estimates in the second step, showing the validity of (CR) up to an extra factor of N, i.e., that the bound is indeed of the order claimed in (5). We formulate the rewriting procedure in terms of the graph $(\Gamma_k, C(\pi))$ first and then give the corresponding formulas for the 2-cumulants that are involved. Note that rewriting is not required if the term does not involve summation over internal indices (see Example 3).

Step 1: Rewriting
Consider $(\Gamma_k, C(\pi))$ as defined above. The rewriting procedure corresponds to carrying out the following reduction algorithm on the graph (see Fig. 3).

Step I Check the set $C(\pi)$ for pairings that involve adjacent edges. If there are any, go through them in the order they appear in $C(\pi)$, replace the corresponding edges of Γ_k and their common vertex by a single edge and remove the pairing from $C(\pi)$. To distinguish the new edges from the edges of the original graph, we assign them the weight one, while any remaining paired edges are assigned the weight zero. The graph resulting from this step is characterized by the property that adjacent edges belong to different pairings.

Step II Check the new graph for any edges of nonzero weight that are adjacent. If there are any, identify the edges that involve the vertex a_l for the smallest value $l \in \{1, \ldots, k\}$, replace the two edges and their common vertex by a single edge and assign it the sum of the weights of the edges that were replaced. This step is then repeated until any edge of the resulting graph with nonzero weight is only adjacent to paired edges, i.e., edges that were assigned the weight zero.

Step III Check the new graph for any edges assigned a nonzero weight, say $w_j > 0$, that are adjacent to two edges belonging to the same pairing, say $(e_j, \varphi(e_j))$. If there are any, go through them in increasing order of the corresponding j and replace the three edges and the two vertices between them by a single edge. After going around the graph once, remove the pairings that were replaced from $C(\pi)$ and assign the new edges the respective weights $w_j + 1$. As this may generate new subgraphs of the same structure, the step is repeated until every edge assigned a nonzero weight is either adjacent to two edges belonging to different pairings or another edge of nonzero weight.

Step IV Repeat Steps II and III until neither can be carried out any further. The graph resulting from this procedure is characterized by the property that whenever two edges are adjacent to an edge assigned a nonzero weight, they are paired edges that belong to different pairings.

Note that the procedure may remove all pairings from $C(\pi)$, which happens if the pairing π was non-crossing. In this case only one vertex connected to itself by an edge (loop) of weight $k/2$ remains, as the weight assigned to an edge reflects the number of pairings removed from $C(\pi)$ in obtaining it. In the general case, we denote the new weighted graph on $k' \leq k$ edges by $\widetilde{\Gamma}_{k'}(\pi)$ and the set of pairings remaining after the above algorithm has been carried out by $\widetilde{C}(\pi) \subseteq C(\pi)$ to obtain a tuple $(\widetilde{\Gamma}_{k'}(\pi), \widetilde{C}(\pi))$. Observe that in the simplification process, connected subgraphs of Γ_k for which the pairings in $C(\pi)$ do not cross are replaced by edges with nonzero weight while the crossing pairings of the original graph remain in $\widetilde{C}(\pi)$. We demonstrate the algorithm for the example given on the right of Fig. 2. For simplicity, the vertex labels and edge weights that are equal to zero are left out.

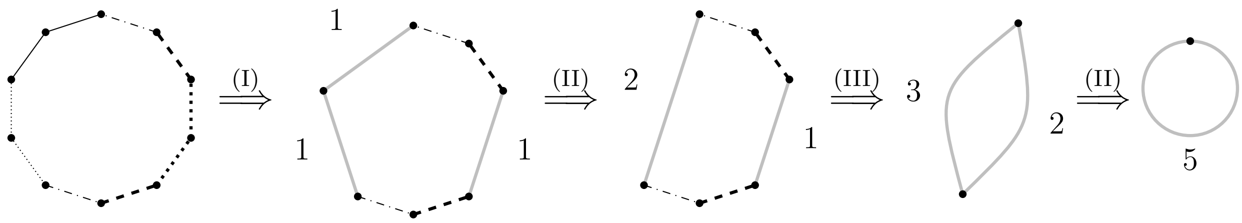

Fig. 3. The steps of the reduction algorithm with arrows indicating the steps.

Next, we explain the algebraic counterpart to the graph reduction on the left-hand side of (5). Step I corresponds to carrying out all summations over internal indices in the 2-cumulants explicitly and introducing matrix elements of T as defined in (9). Hence, every new edge of weight one corresponds to one matrix element of T introduced in the rewriting. The vertex that is removed from the original graph matches the index over which the summation was carried out. Recalling that $\|T\| \leq CN$, the weight of the edge equals the power of N in the bound of the corresponding matrix. This rule will hold more generally along the rewriting procedure, as every edge with nonzero weight w will be represented by a matrix M with $\|M\| \leq N^w$.

Step II corresponds to evaluating summations of the form $\sum_b (M_1)_{a,b}(M_2)_{b,c}$ by introducing elements of the product of the respective matrices. Here, M_1 and M_2 denote any two matrices corresponding to edges of nonzero weight that have been obtained in the rewriting procedure previously, $a, b, c \in \{a_1, \dots, a_k\}$ with $b \neq a$ and $b \neq c$ denote three summation indices. Moreover, $M_1 M_2$ represents the matrix corresponding to the new edge and, as $\|M_1 M_2\| \leq \|M_1\| \, \|M_2\|$, we obtain an explicit norm bound. Hence, adding the weights of the two edges representing M_1 and M_2 on the graph matches the power of N obtained when the bounds for the respective matrix norms are multiplied.

Step III corresponds to carrying out two more summations on the left-hand side of (5) to replace one matrix element and one 2-cumulant by a suitable new matrix element. Let M denote the matrix corresponding to the edge of nonzero weight connecting the $(l+1)$-th and m-th vertices, $l, m \in \{1, \dots, k\}$, then

$$\sum_{a_{l+1}, a_m} M_{a_{l+1}, a_m} \kappa^{(2)}(a_l a_{l+1}, a_m, a_{m+1}) =: \widetilde{M}_{a_l, a_{m+1}}, \tag{15}$$

where the norm of the new matrix can be estimated by $\|\widetilde{M}\| \leq N\|M\|$ following an argument similar to (12). The additional N-power in the bound of the matrix norm again matches the weight of the corresponding new edge assigned in Step III of the algorithm. Therefore, the power of N in the bound for the norm of any matrix M obtained in the procedure coincides with the respective weights assigned to the corresponding edge introduced in Steps II and III. As a result of the rewriting, the term on the left-hand side of (5) is reduced to a summation over a product of 2-cumulants and matrix elements. The power of N in the norm bounds of the respective matrices is readily obtained from $\widetilde{\Gamma}_{k'}(\pi)$, while any vertices remaining in the weighted graph correspond to a summation over the respective index that has not been removed. Further, the pairings of the remaining 2-cumulants have no non-crossing subset.

Step 2: Estimates via summing-in steps

After Step 1, summations over a certain number $k' \leq k$ of indices $a_{l_1}, \dots, a_{l_{k'}}$ satisfying $l_1 < \dots < l_{k'}$, remain. We rename them as $b_1 = a_{l_1}, b_2 = a_{l_2}, \dots, b_{k'} = a_{l_{k'}}$. Recall that Step IV of the reduction algorithm terminates in one of two possible states: either $\widetilde{C}(\pi) = \emptyset$

or in $(\widetilde{\Gamma}_{k'}(\pi), \widetilde{C}(\pi))$ any two edges adjacent to an edge with nonzero weight belong to different pairings.

Assume first that Step 1 reduces the graph $(\Gamma_k, C(\pi))$ to a single vertex b_1 connected to itself by an edge of weight $k/2$. Recall that this full reduction occurs whenever π is non-crossing. The corresponding term involving 2-cumulants thus equals

$$\sum_{b_1} M_{b_1,b_1} = \operatorname{tr}(M),$$

where M denotes the matrix corresponding to the loop in $\widetilde{\Gamma}_{k'}$. By Step 1, $\|M\| \le CN^{k/2}$, giving $|\operatorname{tr}(M)| \le CN^{k/2+1}$. Hence, this term gives a leading contribution to (4).

In the second case, i.e., whenever π is crossing, at least two pairings remain in $\widetilde{C}(\pi)$. If $M^{[j]}$ denotes a matrix that corresponds to an edge of weight j in $\widetilde{\Gamma}_{k'}$, then $\|M^{[j]}\| \le CN^j$. Hence, by defining

$$\left(\mathbf{x}^{[M,j]b_l}\right)_{b_{l+1}} := \frac{1}{N^j} M^{[j]}_{b_l,b_{l+1}}, \ b_{l+1} = 1, \dots, N \tag{16}$$

with $l \in \{1, \dots, k'\}$ and $k' + 1 = 1$, we obtain a vector that satisfies

$$\|\mathbf{x}^{[M,j]b_l}\|_2 \le C \tag{17}$$

uniformly for any choice of b_l. Next, we sum in the vectors using the notation of Lemma 4. Going through the newly introduced vectors (16) in increasing order of the corresponding l, we can replace some of the indices in the remaining 2-cumulants by vectors by carrying out the summations over the corresponding index b_{l+1} explicitly. Since all vectors are defined from the rows of the respective matrices in (16), summing over the corresponding b_{l+1} implies that the sum-in procedure always involves the first index of an index pair, i.e., at most two indices in each 2-cumulant are replaced by a vector. Moreover, the result neither involves summation over internal indices nor, as a consequence of Step III, an index that occurs in the same 2-cumulant both as a superscript of a vector and an argument. In Fig. 4 we visualize the rewriting and summing in procedure for the term on the left of Fig. 2. The small arrows in Fig. 4 point to the index that will be summed in. Again, edge weights that are equal to zero are left out.

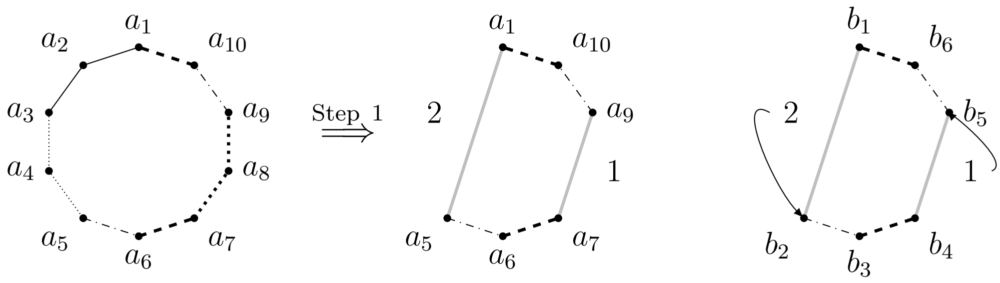

Fig. 4. Visualization of the rewriting and summing in for a crossing partition.

The term obtained after the sum-in procedure is now tractable by recursive summation as illustrated in Examples 3 and 5. We give the algorithm below.

Step i Identify the index b_l in the sum with the smallest subscript l. Whenever the index occurs both as a superscript of a vector and an argument, start the procedure with the 2-cumulant that involves b_l as an argument. Otherwise, use an estimate similar to (6) with a_1 replaced by b_l and start the procedure with the 2-cumulant that involves b_l.

Step ii Isolate the chosen 2-cumulant and perform the summation over the indices it involves similar to (14). Whenever an index occurs both as a superscript of a vector and an argument in this step, carry out the summation for the 2-cumulant that involves the index as an argument and use the uniform bound (17) for the vector.

Step iii Repeat Step ii, i.e., identify the index b_l in the remaining sum with the smallest l and carry out Step ii for the 2-cumulant that involves b_l as an argument until all summations have been performed.

Note in particular that the bounds obtained from Lemmas 2 and 4 in Step ii follow (CR).

It remains to collect the factors N^j resulting from the normalization (16). To check the validity of (CR), we compare the number of summations that were carried out and the order of the bound that is obtained in exchange. First, observe that introducing each matrix element of T^j requires carrying out summations over $2j-1$ consecutive indices while $\|T^j\| \leq CN^j$. Hence, these matrix elements yield a factor $\sqrt{N}$ more than prescribed by (CR) if estimated trivially. Moreover, carrying out a rewriting step corresponding to Step II or III on the graph does not change this fact. Indeed, if the matrix elements of M_1 and M_2 are obtained from carrying out $2j_1-1$ and $2j_2-1$ summations and the matrices are bounded by CN^{j_1} and CN^{j_2} in norm, respectively, then the matrix elements of M_1M_2 account for $2(j_1+j_2)-1$ summations while the matrix itself obeys a norm bound of order $N^{j_1+j_2}$. Similarly, if the matrix elements of M are obtained from carrying out $2j-1$ summations and $\|M\| \leq CN^j$, then the matrix elements of $\widetilde{M}$ in (15) account for two more summations, i.e., $2(j+1)-1$ in total, while the power of N in the norm bound is increased by one. Since along the sum-in procedure one additional summation is carried out per matrix to replace an index by a vector of the form (16) after the rewriting of the term, the additional factor of $\sqrt{N}$ per matrix is balanced out such that all estimates obtained subject to (CR).

Hence, at most $k+1$ powers of $\sqrt{N}$ are collected from bounding the k summations with the extra factor $\sqrt{N}$ being obtained whenever an estimate similar to (6) is used to start the recursive summation procedure. Thus, the bound is of order $N^{-1/2}$ here, i.e., stronger than (5), showing that all crossing pairings are subleading. This completes the proof of (5).

3 Proof of (3) in the General Case

Let now $k \geq 1$ be arbitrary. We aim to show that

$$N^{-k/2-1}\Big|\sum_{a_1,\dots,a_k}\prod_{B\in\pi}\kappa^{(|B|)}(a_ja_{j+1}|j\in B)\Big| \leq C(k) \tag{18}$$

for all $\pi \in \Pi_k$. This would complete the proof of (3) via (4). We start by proving the necessary bounds for j-cumulants with $j \geq 3$ and give the proof of (18) in Section 3.1. Note that no internal index is summed up below.

Lemma 6. *Assume that* (A3) *holds. Then*

$$\sum_{a_1,a_2,a_3,a_4}|\kappa^{(3)}(a_1a_2,a_3a_4,a_5a_6)| \leq C, \tag{19}$$

$$\sum_{a_1,\dots,a_5}|\kappa^{(3)}(a_1a_2,a_3a_4,a_5a_6)| \leq CN, \tag{20}$$

$$\sum_{a_1,\dots,a_6}|\kappa^{(3)}(a_1a_2,a_3a_4,a_5a_6)| \leq CN^2 \tag{21}$$

uniformly for any choice of the unsummed indices. In particular, the estimates follow (CR).

Proof. Applying (2) to the 3-cumulant in (19) and taking the maximum in a suitable way in the resulting terms, we obtain, e.g.,

$$\sum_{a_1,\dots,a_4} |\kappa^{(2)}(a_1a_2, a_3a_4)\kappa^{(2)}(a_3a_4, a_5a_6)| \leq \Big(\max_{a_3,a_4} \sum_{a_1,a_2} |\kappa^{(2)}(a_1a_2, a_3a_4)|\Big) \sum_{a_3,a_4} |\kappa^{(2)}(a_3a_4, a_5a_6)|,$$

as well as similar bounds for the terms corresponding to the other possible spanning trees. Hence, the summation over up to two index pairs is bounded by a constant, giving (19). The remaining estimates (20) and (21) readily follow, as the additional summations yield a factor of N or N^2, respectively. □

Observe that the bounds given in (19) and (20) imply that, e.g.,

$$\sum_{a_2} |\kappa^{(3)}(a_1a_2, a_2a_3, a_4a_5)| \leq \sum_{a_2,a_2'} |\kappa^{(3)}(a_1a_2, a_2'a_3, a_4a_5)| \leq C \leq C\sqrt{N} \tag{22}$$

uniformly for any choice of the unsummed indices. In particular, the estimates comply with (CR) and there is no need to perform summations over internal indices separately as in Section 2. The only exception occurs for $k = 3$, where one obtains

$$\sum_{a_1,a_2,a_3} |\kappa^{(3)}(a_1a_2, a_2a_3, a_3a_1)| \leq \sum_{a_1,a_2,a_3,a_1',a_2',a_3'} |\kappa^{(3)}(a_1a_2, a_2'a_3, a_3'a_1')| \leq CN^2 \tag{23}$$

instead of the bound of order $N^{3/2}$ prescribed by the counting rule. As (23) is the only term for $k = 3$ due to (A1), and $k/2 + 1 = 5/2$, it follows that (3) holds for $k = 3$. Hence, we can exclude the $k = 3$ case from the following analysis and assume that all estimates for 3-cumulants with or without internal indices comply with (CR). Similarly to Lemma 4, we define

$$\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6) := \sum_{a_1} \kappa^{(3)}(a_1a_2, a_3a_4, a_5a_6)x_{a_1} \tag{24}$$

for $\mathbf{x} \in \mathbb{R}^N$ with terms such as $\kappa^{(3)}(\mathbf{x}a_2, a_3\mathbf{y}, a_5a_6)$ being defined in the same way. Again, we follow the convention that vectors only occur as the first index of every index pair.

Lemma 7. *Assume that* (A3) *holds and let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^N$. *Then the summation over any number of indices of* $|\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6)|$, $|\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, a_5a_6)|$, *or* $|\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, \mathbf{z}a_6)|$ *satisfies* (CR). *Thus, for one vector we, e.g., have*

$$\sum_{a_2} |\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6)| \leq CN^{1/2}\|\mathbf{x}\|_2, \tag{25}$$

$$\sum_{a_2,a_4} |\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6)| \leq CN\|\mathbf{x}\|_2 \tag{26}$$

uniformly for any choice of the unsummed indices. Similar bounds hold if two or three vectors are involved, respectively, e.g.,

$$\sum_{a_5} |\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, a_5a_6)| \leq CN^{1/2}\|\mathbf{x}\|_2\|\mathbf{y}\|_2, \tag{27}$$

$$\sum_{a_2} |\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, \mathbf{z}a_6)| \leq CN^{1/2}\|\mathbf{x}\|_2\|\mathbf{y}\|_2\|\mathbf{z}\|_2.$$

Moreover, we have bounds that are stronger than (CR) *by a factor of* $\sqrt{N}$ *for every summation that is carried out over two consecutive indices belonging to different index pairs, e.g.,*

$$\sum_{a_4,a_5} |\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6)| \leq CN^{1/2}\|\mathbf{x}\|_2. \tag{28}$$

$$\sum_{a_2,\ldots,a_6} |\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6)| \leq CN^{3/2}\|\mathbf{x}\|_2, \tag{29}$$

$$\sum_{a_4,a_5} |\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, a_5a_6)| \leq CN^{1/2}\|\mathbf{x}\|_2\|\mathbf{y}\|_2. \tag{30}$$

In particular, for 3-cumulants, a summation over an internal index also contributes at most a factor of $\sqrt{N}$.

Proof. The proof is divided into three general arguments. First, as the bounds obtained in Lemma 6 are stronger than (CR), estimating the vector elements by $\max_j |v_j| \leq \|\mathbf{v}\|_2$ for $\mathbf{v} = \mathbf{x}, \mathbf{y}, \mathbf{z}$, respectively, and performing the summation directly yields the desired estimate in most cases. Whenever this argument does not yield a sufficiently strong bound, we apply the Cauchy-Schwarz inequality similar to the proof of Lemma 4. Lastly, it remains to check that the estimates obtained are strong enough to satisfy the second part of the lemma and also comply with (CR) if summation over internal indices is included. As the latter can be treated by introducing additional summation labels similar to (22), it is enough to derive suitable bounds for distinct $a_1, \ldots, a_6$.

Assume first that the 3-cumulant involves only one vector. Here, estimating $|x_j| \leq \|\mathbf{x}\|_2$ and applying (19) yields

$$\sum_{a_2,a_3,a_4} |\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6)| \leq \|\mathbf{x}\|_2 \sum_{a_1,\ldots,a_4} |\kappa^{(3)}(a_1a_2, a_3a_4, a_5a_6)| \leq C\|\mathbf{x}\|_2, \tag{31}$$

which immediately implies (25) and (26). The same N-independent bound as (31) holds if one sums over a_2, a_5, a_6 instead. Whenever all three index pairs are involved in the summation, Lemma 6 implies a bound of order N, i.e.,

$$\sum_{a_2,\ldots,a_5} |\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6)| \leq \|\mathbf{x}\|_2 \sum_{a_1,\ldots,a_5} |\kappa^{(3)}(a_1a_2, a_3a_4, a_5a_6)| \leq CN\|\mathbf{x}\|_2, \tag{32}$$

and the same estimate holds if the summation over some other index $a_2, \ldots, a_5$ is left out instead of a_6. In particular, all summations over two, three and four distinct indices yield a bound of at most order N, which complies with (CR). The validity of (CR) for summation over five distinct indices follows from (21).

Now we prove (28) and (29) by applying (2) and estimating one 2-cumulant in the bound similar to (7) in the proof of Lemma 4. This leads to

$$\begin{aligned}
&\sum_{a_1,a_3,a_5} |x_{a_1}\kappa^{(2)}(a_1a_2, a_3a_4)\kappa^{(2)}(a_3a_4, a_5a_6)| \\
&\leq \Big(\max_{a_3} \sum_{a_5} |\kappa^{(2)}(a_3a_4, a_5a_6)|\Big) \sum_{a_1,a_3} |x_{a_1}\kappa^{(2)}(a_1a_2, a_3a_4)| \leq CN^{1/2}\|\mathbf{x}\|_2,
\end{aligned}$$

which, together with similar estimates obtained for the other possible spanning trees on the complete graph on three vertices, proves (28). The estimate in (29) follows similarly.

Applying (22) and (32) implies the remaining stronger bounds for the second part of the lemma, which completes the proof for $\kappa^{(3)}(\mathbf{x}a_2, a_3a_4, a_5a_6)$.

Next, assume that two vectors are involved in $\kappa^{(3)}$. Here, estimating $|x_j| \leq \|\mathbf{x}\|_2$ and $|y_j| \leq \|\mathbf{y}\|_2$, applying (19) yields

$$\sum_{a_2,a_4} |\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, a_5a_6)| \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2 \sum_{a_1,\ldots,a_4} |\kappa^{(3)}(a_1a_2, a_3a_4, a_5a_6)| \leq C\|\mathbf{x}\|_2\|\mathbf{y}\|_2. \tag{33}$$

Further, arguing similarly to (7) gives

$$\sum_{a_1,a_3,a_5} |x_{a_1}y_{a_3}\kappa^{(2)}(a_3a_4, a_5a_6)\kappa^{(2)}(a_5a_6, a_1a_2)| \leq CN^{1/2}\|\mathbf{x}\|_2\|\mathbf{y}\|_2,$$

which, together with bounds of at most order $N^{1/2}$ for the terms corresponding to the other possible spanning trees, implies (27). An analogous estimate holds for summation over a_6, showing the validity of (CR) for a single summation. The remaining cases for summation over distinct indices follow similar to (33) by applying (20) and (21).

Note, however, that the above bounds are again not sufficient to imply (CR) for summation over internal indices. Arguing as in the proof of Lemma 4 yields the stronger estimate

$$\sum_{a_1,a_3,a_4,a_5} |x_{a_1}y_{a_3}\kappa^{(2)}(a_1a_2, a_3a_4)\kappa^{(2)}(a_3a_4, a_5a_6)| \leq CN^{1/2}\|\mathbf{x}\|_2\|\mathbf{y}\|_2.$$

Again, one can estimate similarly for the terms corresponding to the other possible spanning trees, which implies (30). Lastly, one needs to check that also

$$\sum_{a_2,a_4,a_5} |\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, a_4a_5)| \leq \sum_{a_2,a_4,a_5,a_4'} |\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, a_4'a_5)| \leq CN^{3/2}\|\mathbf{x}\|_2\|\mathbf{y}\|_2,$$

which follows from a similar argument. This completes the proof for $\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, a_5a_6)$.

For 3-cumulants that involve three vectors, note that writing out the term similar to (24) already involves all three index pairs. Whenever two summations are carried out, the estimates obtained from Lemma 6 comply with (CR), e.g.,

$$\sum_{a_2,a_4} |\kappa^{(3)}(\mathbf{x}a_2, \mathbf{y}a_4, \mathbf{z}a_6)| \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2\|\mathbf{z}\|_2 \sum_{a_1,\ldots,a_5} |\kappa^{(3)}(a_1a_2, a_3a_4, a_5a_6)| \leq CN\|\mathbf{x}\|_2\|\mathbf{y}\|_2\|\mathbf{z}\|_2,$$

but a different argument is needed for estimating zero, one, or three summations. Here, arguing as in the proof of Lemma 4 gives

$$\begin{aligned}
&\sum_{a_2}\sum_{a_1,a_3,a_5} |x_{a_1}y_{a_3}z_{a_5}\kappa^{(2)}(a_1a_2, a_3a_4)\kappa^{(2)}(a_3a_4, a_5a_6)| \\
&\leq \|\mathbf{z}\|_2\Big(\max_{a_3}\sum_{a_5} |\kappa^{(2)}(a_3a_4, a_5a_6)|\Big) \sum_{a_1,a_2,a_3} |x_{a_1}y_{a_3}\kappa^{(2)}(a_1a_2, a_3a_4)| \leq C\|\mathbf{x}\|_2\|\mathbf{y}\|_2\|\mathbf{z}\|_2
\end{aligned}$$

together with similar N-independent bounds for the terms corresponding to the other possible spanning trees. By a similar argument, we obtain a bound of order N for three summations. As internal indices cannot occur here, the proof of the lemma is complete. □

Lastly, we derive the necessary estimates to incorporate cumulants of order four and higher. As no exceptions similar to (23) occur, we directly include summations over internal indices into the bound.

Lemma 8. *Assume that* (A3) *holds and let* $j \geq 4$. *Then*

$$\sum_{a_1,\ldots,a_{2j}} |\kappa^{(j)}(a_1 a_2, \ldots, a_{2j-1} a_{2j})| \leq CN^2.$$

Proof. After applying (2), bound the term by considering all possible spanning trees on the complete graph on j vertices. Starting the summation at the leaves of the respective tree, sum up one index pair of each 2-cumulant. Continuing the procedure until the root of the tree is reached, an N-independent bound is obtained in every step. Finally, a summation over all four indices of the last 2-cumulant remains, which yields the claimed bound of order N^2 by applying the last estimate of Lemma 2. □

Note that Lemma 8 in particular implies that

$$\sum_{a_1,\ldots,a_j} |\kappa^{(j)}(a_1 a_2, a_2 a_3, a_3 a_4, \ldots, a_j a_1)| \leq \sum_{a_1,\ldots,a_j,a'_1,\ldots,a'_j} |\kappa^{(j)}(a_1 a_2, a'_2 a_3, \ldots, a'_j a'_1)| \leq CN^2$$

for any $j \geq 4$ such that no special treatment is needed for (CR). Moreover, whenever $j \geq 5$, the bound is stronger than (CR). However, Lemma 8 only gives a bound of N^2 independent of the number of summations carried out.

3.1 Including 3-cumulants

Building on the proof of (5), assume next that $\pi \in \Pi_k$ is chosen such that $|B| \in \{2, 3\}$ for all $B \in \pi$. Here, we obtain the following bound.

Lemma 9. *Under assumptions* (A1)-(A3), *let* $k \geq 1$ *and choose a partition* $\pi \in \Pi_k$ *with* $|B| \in \{2, 3\}$ *for all* $B \in \pi$. *Then*

$$N^{-k/2-1} \sum_{a_1,\ldots,a_k} \prod_{B \in \pi} |\kappa^{(|B|)}(a_j a_{j+1} | j \in B)| \leq C(k). \tag{34}$$

Proof. Excluding the case where $|B| = 2$ for all $B \in \pi$, i.e., (5), assume that the partition involves at least one set with three elements. To estimate the left-hand side of (34), we extend the argument used in the proof of (5) for crossing partitions. The first step is to sum up internal indices. Since the summation over internal indices in 3-cumulants does not interfere with (CR), only the 2-cumulants that involve internal indices are considered here. Next, we sum in the matrix elements that were obtained from the rewriting and, in the third step, estimate the summation that remains.

Step 1: Rewriting
Consider the graph Γ_k and the (ordered) set $E_k = \{(a_1, a_2), \ldots, (a_k, a_1)\}$. Similarly to the proof of (5), we go through its elements one by one to extract an (ordered) set $C(\pi)$ of groupings in which each element of E_k occurs exactly once. The elements of $C(\pi)$ are pairs and 3-tuples from E_k corresponding to the 2- and 3-cumulants. By restricting the algorithm from the proof of (5) to the pairs in $C(\pi)$, extract a weighted graph $(\widetilde{\Gamma}_{k'}(\pi), \widetilde{C}(\pi))$ on $k' \leq k$ vertices that corresponds to the term obtained from rewriting all summations over internal

indices in the 2-cumulants. Recall that the norm bounds for any matrix resulting from the rewriting procedure are recorded in $\widetilde{\Gamma}_{k'}(\pi)$ as the weights assigned to the edges. We rename the remaining k' summation indices as $b_1, \ldots, b_{k'}$. The resulting graph $(\widetilde{\Gamma}_{k'}(\pi), \widetilde{C}(\pi))$ has the property that any two edges adjacent to an edge with nonzero weight do not belong to the same pair. However, they may still belong to the same 3-tuple, as 3-tuples directly transfer from $C(\pi)$ to $\widetilde{C}(\pi)$.

Step 2: Sum-In
Similar to the proof of (5), we aim to rewrite the matrix elements obtained in Step 1 into vectors of the form (16) and perform a sum-in procedure to replace some of the indices with vectors in the remaining 2- and 3-cumulants. Consider the possible subgraphs in which an edge with nonzero weight is adjacent to two edges that belong to the same 3-tuple. Here, rewriting and estimating the term using the vectors in (16) fails similarly to the second part of Example 5 such that the corresponding terms have to be treated differently. We visualize the respective subgraphs in Fig. 5 below, where edges belonging to the same 3-tuple are indicated by the same color and w_j (resp. w_{j_1}, w_{j_2}) denotes some nonzero edge weight. For simplicity, the labels of the individual vertices and edge weights that are equal to zero are left out and the remainder of the graph is indicated by horizontal dots.

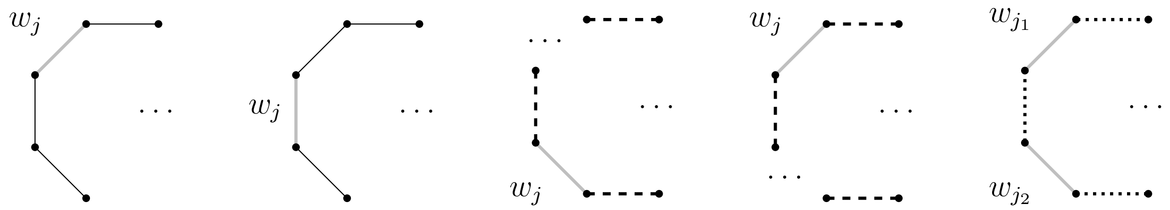

Fig. 5. The three types of subgraphs to be considered separately.

Next, we gather the 3-cumulant and the matrix element(s) corresponding to the respective subgraphs in Fig. 5 into one term that we refer to as type 1 (solid), type 2 (dashed) or type 3 (dotted), respectively. Note that the two solid and two dashed graphs look very similar, but are still not identical since the cyclic ordering breaks the symmetry. As a result of Step III of the algorithm used in the proof of (5), any matrix element that does not occur in a term of type 1, 2 or 3 can be treated using the same sum-in procedure as in the proof of (5), i.e., by applying (16) and performing the summation over the corresponding b_{l+1} explicitly. Again, defining the vectors by the rows of the respective matrices ensures that only the first index of every index pair in the remaining 2- and 3-cumulants may be replaced by a vector. At this point, any separate matrix element occurs in a term of type 1, 2 or 3. Note that also up to one (types 1 and 3) or two (type 2) indices of the 3-cumulant in the respective terms may have been replaced by a vector along the sum-in procedure.

Step 3: Estimates
To estimate the remaining summations, we apply a recursive summation procedure similar to the proof of (5). Recall that the summation over any number of indices of a 2- or 3-cumulant follows (CR) by Lemmas 2, 4, 6, and 7. This includes in particular summations over internal indices in the 3-cumulants and any summation over internal indices of 2-cumulants resulting in the sum-in of the vectors in (16). Hence, it remains to show that summations over terms of type 1, 2, and 3, respectively, comply with (CR). Here, we estimate the matrix element or elements trivially by the norm of the corresponding matrix. This yields a factor of $\sqrt{N}$ more than prescribed by the counting rule, which is compensated by applying the stronger bounds from Lemma 6 or 7, respectively, for the 3-cumulant. We give the details below.

Assume first that the summation on the left-hand side of (18) reduces to one single term of type 1, with the internal index occurring, e.g., at b_4, or of type 3. We obtain from Lemma 6 and (22) that

$$\sum_{b_1,\dots,b_5} |\kappa^{(3)}(b_1b_2, b_3b_4, b_4b_5)| \le CN^2, \qquad \sum_{b_1,\dots,b_6} |\kappa^{(3)}(b_1b_2, b_3b_4, b_5b_6)| \le CN^2.$$

Observe that these estimates are better than the bound prescribed by (CR) by one or two factors of $\sqrt{N}$, respectively. Hence, the final bound for the term follows (CR) again.

Whenever a term of type 1, 2, or 3 is encountered along the recursive summation procedure, we argue similarly. Recall that Step ii used in the proof of (5) requires summing over all remaining indices for every 2-cumulant that is encountered. For terms of type 1, 2, or 3, we modify this step and sum up all but one (type 1 and 3) or two (type 2) indices instead, always leaving the index b_l for the largest value of l in each connected subgraph unsummed. The index or indices that are left out always belong to another 2- or 3-cumulant or term of type 1, 2, or 3 and are thus considered in later steps of the procedure. The summation rule is visualized in Fig. 6 below, where the large dots denote the indices that will be summed up after estimating the matrix element(s) trivially.

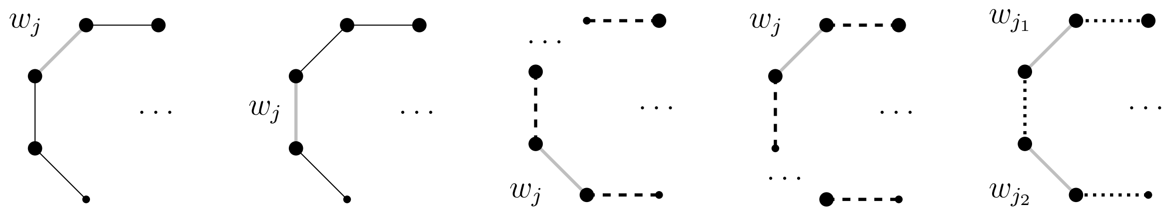

Fig. 6. The summation rule for terms of type 1, 2 and 3.

For the terms of type 1 or 3, leaving one index of the respective 3-cumulant unsummed always allows for a bound of order N by Lemma 6 or (32), i.e., at least $(\sqrt{N})^2$ better than (CR) would give, which compensates for the additional factor of $\sqrt{N}$ or N from estimating the matrix elements, respectively. The same holds for terms of type 2. Note, however, that due to the dashed subgraphs in Fig. 6 having two connected components, we may also encounter the case that only two summations remain. Here, we apply (19), (28) or (30) whenever the 3-cumulant involves zero, one or two vectors, respectively. This allows to estimate the remaining two summations by a bound of at most order $N^{1/2}$, showing that (CR) holds again. Hence, we can apply a similar recursive summation procedure as used in the proof of (5) given by the following modified algorithm.

Step i Identify the summation index b_l occurring in a term of type 1, 2 or 3 that corresponds to the smallest subscript l. If there are no such terms, carry out Step i as in the proof of (5), possibly choosing an index occuring in a 3-cumulant.

Step ii If the chosen index occurs in a term of type 1, 2 or 3, estimate the corresponding matrix element(s) using the bound for the matrix norm, then isolate the corresponding 3-cumulant and perform the summation according to Fig. 6. Otherwise, carry out Step ii as in the proof of (5), possibly applying Lemma 6 or 7 if the index chosen in Step i occurs in a 3-cumulant.

Step iii Identify the index b_l in the remaining sum that corresponds to the smallest subscript l and carry out Step ii until all summations have been evaluated.

Note that starting the summation with estimating a matrix element breaks the cyclic structure of the graph similarly to (6). Further, we have shown that all estimates obtained in Step ii follow (CR). The final bound for the sum obtained at the end of the procedure is thus at most of order $N^{k/2+1/2}$, showing that we obtain a sub-leading contribution to (4) if the term on the left-hand side of (18) involves 3-cumulants. $\square$

3.2 Including Cumulants of Order Four and Higher

With the necessary tools established, we proceed to estimating the summation (18) in the general case, i.e., when we have cumulants (tuples) of arbitrary order.

Proof of (18). Let $k \in \mathbb{N}$ and $\pi \in \Pi_k$ be arbitrary. Excluding the cases already considered, we can assume that $k \geq 4$ and that the partition includes at least one set of four or more elements. As the term on the left-hand side of (18) may involve summations over internal indices in 2-cumulants, we first follow Step 1 of the proof of Lemma 9, treating j-cumulants for $j > 3$ similar to the 3-cumulants. The term resulting from this procedure is a product of cumulants of order three and higher, 2-cumulants that do not involve internal indices and the matrix elements corresponding to the edges of nonzero weight in $(\widetilde{\Gamma}_{k'}(\pi), \widetilde{C}(\pi))$. Again, we rename the indices remaining after the rewriting step as $b_1, \ldots, b_{k'}$. Next, we follow Step 2 of the proof of Lemma 9 to incorporate some of the matrix elements into the remaining 2- and 3-cumulants by applying the sum-in procedure. Further, any matrix elements occurring in a term of type 1, 2 or 3 (see Fig. 5) are gathered together with the corresponding 3-cumulant. Assume first that the partition π is chosen such that all matrix elements remaining after the sum-in procedure are associated with a term of type 1, 2 or 3.

Recall that the counting rule established for 2- and 3-cumulants prescribes that every individual summation yields a factor of $\sqrt{N}$, while the estimate for higher-order cumulants yields a contribution of N^2 when summing over all indices in the respective cumulant. To apply both rules simultaneously, we divide the terms obtained from the rewriting procedure into two factors F and G, where G collects the 2- and 3-cumulants, as well as the terms of type 1, 2 and 3 and the additional factors of N obtained from the N^{-j} normalization in (16), while the higher-order cumulants constitute F. Whenever no rewriting is required, we can split the left-hand side of (18) directly into

$$F(b_1, \ldots, b_k) := \prod_{\substack{B \in \pi \\ |B| \geq 4}} |\kappa^{(|B|)}(b_j b_{j+1} | j \in B)|, \quad G(b_1, \ldots, b_k) := \prod_{\substack{B \in \pi \\ 2 \leq |B| \leq 3}} |\kappa^{(|B|)}(b_j b_{j+1} | j \in B)|.$$

Next, consider the set $X := \{j : b_j \text{ appears in a cumulant that belongs to } F\}$. Abbreviating $b_X = \{b_j : j \in X\}$ and $b_{X^c} = \{b_1, \ldots, b_{k'}\} \setminus b_X$, it follows that

$$\sum_{b_1, \ldots, b_{k'}} F(b_X) G(b_1, \ldots, b_{k'}) \leq \Big(\sum_{b_X} F(b_X) \Big) \Big(\max_{b_X} \sum_{b_{X^c}} G(b_1, \ldots, b_{k'}) \Big)$$

by taking the maximum over all indices in b_X in the factors that occur in G.

Let $n \geq 1$ be the number of factors in F. Since every cumulant included in F is of order four or higher, each factor involves at least eight indices as arguments. As every index belongs to two edges and, hence, has to appear exactly twice in the product of F and G, there are at least $4n$ distinct indices in F. By definition, the total number of distinct indices occurring in F is equal to $|X|$ such that $|X| \geq 4n$. Introducing additional summation labels $b'_1, \ldots, b'_l$ to sum over all indices involved in the respective factors and applying Lemma 8 thus implies

$$\sum_{b_X} F(b_X) \leq \sum_{b_X, b'_1, \ldots, b'_l} F(b_X, b'_1, \ldots, b'_l) \leq C(N^2)^n \leq CN^{|X|/2}. \tag{35}$$

For the summation involving G, we follow Step 3 from the proof of Lemma 9. This yields

$$\max_{b_X} \sum_{b_X^c} G(b_1, \ldots, b_{k'}) \leq CN^{(k-|X|)/2+1/2} \tag{36}$$

from applying the recursive summation procedure, as all terms in the sum follow (CR). Hence, we obtain a contribution of order $\sqrt{N}$ for every summation, including the ones that were carried out along the rewriting and sum-in step, and possibly an additional factor $\sqrt{N}$ from an estimate similar to (6). Combining (35) and (36) thus yields

$$N^{-k/2-1} \Big(\sum_{b_X} F(b_X) \Big) \Big(\max_{b_X} \sum_{b_X^c} G(b_1, \ldots, b_{k'}) \Big) \leq CN^{-1/2}.$$

In the previously excluded cases, there either is a term similar to the terms of type 1, 2 or 3 with the 3-cumulant replaced by a cumulant of order four or higher, or a matrix element for which introducing vectors as in (16) would require summing it into a higher-order cumulant. In both cases, we estimate the matrix element trivially by the norm of the corresponding matrix, which yields a factor of $\sqrt{N}$ more than prescribed by (CR). However, the occurrence of the matrix element implies that the j-cumulant must involve at least $j+1$ distinct indices. Hence, one more summation than in the minimal case is carried out when the summation over all indices in the cumulant is evaluated. Hence, the additional factor $\sqrt{N}$ is balanced out. This implies that the estimates again follow (CR), giving the claim in the general case. In particular, we obtain a sub-leading contribution to (4) whenever the term on the left-hand side of (18) involves cumulants of order four or higher. □

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