# An algebraic approach to some models in the KPZ "Universality class"

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18-December-2014

# Contents

**Abstract** **2**

**Generating functions** **4**

**Representations of finite grups** **6**

**Partitions and Young tableaux** **9**

**The ring of symmetric functions** **15**

**Probabilistic models from the "KPZ Class", related to what we introduced so far** **20**

**References** **25**

# Abstract

The goal of this talk is to make the audience familiar with certain algebraic and combinatorial methods which prove useful in the study of the models in the "KPZ Universality class". Not only they prove useful, but some of the models can be seen to arise from purely combinatorial objects. An example for such a model is the Last Passage Percolation, which is known to be in the "KPZ Universality Class". It turns out that the LPP can be constructed via the Viennot's geometric construction of the Robinson-Schensted correspondence, which gives a bijection between the the elements of $S_n$ - the symmetric group and the set of pairs of standard Young tableaux of shape $\lambda \vdash n$. Many other surprising and fascinating connections appear in the study of those models. Those observations (and not only) led to a conjecture that all those models should belong to one "Universal Class". This conjectural class is named after the KPZ stochastic partial differential equation, because the conjecture is that this equation will describe the fluctuations of all the models in the class.

A brief but detailed overview will be provided for the basic properties of the ring of Symmetric functions over $\mathbb{C}$ and for the combinatorics of the Young tableaux. Once the basic definitions and properties are introduced, we shall give examples of different probabilistic models that either arise from or are related to those objects, and that belong to the KPZ Universality Class. The distributions of some of those models will be explicitly given, along with some comments on them.

The final goal of the talk is to point out how the introduced algebraic and combinatorial methods can be used in the computation of the distributions of some models in the KPZ Universality Class and, most of all to use them in order to give a taste of the deep connections between some a priori very different models.

# Introduction

The abbreviation KPZ stands for Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang, who in 1986 introduced a stochastic differential equation (called the KPZ equation):

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t)$$

In their paper, they propose the KPZ equation as a model of the time evolution of some class of surface growth models such as the Eden model (which describes the growth of specific types of clusters such as bacterial colonies and deposition of materials) and study its properties. The proposed equation has been obtained by the stochastic heat equation by adding a non-linear term and Gaussian white noise.

The study of such non-linear stochastic differential equation is a very challenging task. One has to worry about the regularity of the solutions etc. However the equation quickly became the default model for random interface growth in physics.

With the time passing, both mathematicians and physicists were able to relate many different probabilistic models with the KPZ equation (they exhibit similar fluctuations as its solutions). Many fascinating connections appeared between models, which do not seem close at all. It has been conjectured that there is a universality class of models, whose fluctuations would be described by the KPZ equation (The name of this conjectural class is the "KPZ Universality Class"). In the past years this topic has grown hotter and hotter and many fascinating results were obtained. One of the last results was awarded with a Fields medal in 2014, namely the article "Solving the KPZ equation" by Martin Hairer, where he introduces a new method for solving it. This method proves to be much more powerful than the one known until now (the Cole-Hopf transform, which works well for some model but not for other). The following sentence is essentially a quote from the article of Martin Hairer: "In particular, our construction completely bypasses the Cole-Hopf transform, thus laying the groundwork for proving that the KPZ equation describes the fluctuations of systems in the KPZ universality class".

With the time passing deep connections were discovered between some models in the "KPZ universality class" and some purely algebraic and combinatorial object. Along with the analysis tools that are used in the study of the models, some algebraic ones were introduced as well and proved very useful. The goal of the present talk will be essentially to present the basic algebraic and combinatorial tools, that are used in the study of the models in the KPZ Universality Class and moreover to exhibit their fascinating connection with a theory that seems so far away at the first sight.

# Generating functions

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

**Definition 0.1.** *The **Generating Function** corresponding to the sequence* $\{a_n\}_{n=1}^{\infty}$ *is*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

All generating functions are members of the set of *formal* power series over $\mathbb{C}$:

$$\mathbb{C}[[x]] = \{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in \mathbb{C}\,,\, n \geq 0\} \tag{2}$$

$\mathbb{C}[[x]]$ has a ring structure with the standard operations of addition and multiplication of power series.

The word "*formal*" stands for the fact that we do not consider any convergence, but consider the elements of $\mathbb{C}[[x]]$ as algebraic objects. Sometimes we would even "*formally*" differentiate *formal* power series or sum them to "functions", without considering whether the obtained "function" will even exist in terms of convergence. However, for some calculations we would need to assure convergence and would apply additional restrictions on the coefficients $a_n$ in order to ensure it.

A nice way to find a generating function for a given sequence $\{a_n\}_{n=1}^{\infty}$ (far from being the only one) is the following algorithm:

1. Find a set $S$ with a parameter, such that the number of elements in $S$ whose parameter is $n$ would be $a_n$
2. Express the elements of $S$ in terms of "or", "and" and the parameter
3. Translate this expression into a generating function using "+' for "or", "$\times$" for "and" and $x^n$

We shall illustrate the algorithm with two examples, one trivial and one substantial:

**Example 0.1.** *Let* $a_n =$ *number of* $n$*-element subsets of the set* $\{1,2,3\} = \binom{3}{n}$ *, i.e.* $a_0 = 1\,,\, a_1 = 3\,,\, a_2 = 3\,,\, a_3 = 1\,,\, a_n = 0 \quad \forall n \geq 4$
*We follow the algorithm:*

1. *Let* $S = 2^{\{1,2,3\}}$ *- the power set of the three elements set* $\{1,2,3\}$ *and for every* $T \in S$ *,* $n(T) \doteq |T|$
2. $T = (1 \notin T \,\mathit{or}\, 1 \in T) \,\mathit{and}\, (2 \notin T \,\mathit{or}\, 2 \in T) \,\mathit{and}\, (3 \notin T \,\mathit{or}\, 3 \in T)$
3. $n$ *stands for the number of elements of* $T$*, so* $1 \notin T$ *and* $1 \in T$ *would translate to* $x^0$ *and* $x^1$

$$\Rightarrow\ f(x) = (x^0 + x^1)(x^0 + x^1)(x^0 + x^1) = (1+x)^3$$

And now the substantial example:

**Example 0.2.** *Let* $a_n = p(n)$ *, where* $p(n) = \{$*number of partitions (shapes of Young tableaux) of* $n \in \mathbb{N}\}$. *We would like to find a generating function* $\sum_{n\geq 0} p(n)x^n$. *Let* $\lambda \vdash n$ *(*$\lambda$ *be a partition of* $n$*). We have* $\lambda = (\lambda_1^{\mu_1}, \lambda_2^{\mu_2}, \ldots, \lambda_k^{\mu_k})$ *, where* $k \leq n$ *,* $\lambda_i \in \mathbb{N} \cap \{0\}$ *,* $0 \leq \lambda_i \leq n$ *and* $\mu_i \in \mathbb{N}, 0 < \mu_i \leq n$ *and* $\lambda_i^{\mu_i} = (\underbrace{\lambda_i, \lambda_i, \ldots, \lambda_i}_{\mu_i\ times})$. *We have:*

$$\lambda = (1^0 \in \lambda \ \mathit{or}\ 1^1 \in \lambda \ \mathit{or}\ 1^2 \in \lambda \ \mathit{or}\ \ldots)\ \mathit{and}\ (2^0 \in \lambda \ \mathit{or}\ 2^1 \in \lambda \ \mathit{or}\ 2^2 \in \lambda \ \mathit{or}\ \ldots)\ \mathit{and}\ \ldots$$

*which translates to:*

$$f(x) = \underbrace{(x^0 + x^1 + x^{1+1} + x^{1+1+1} + \ldots)}_{formally\ sums\ to\ \frac{1}{1-x}} \underbrace{(x^0 + x^2 + x^{2+2} + x^{2+2+2} + \ldots)}_{formally\ sums\ to\ \frac{1}{1-x^2}}$$

$$\underbrace{(x^0 + x^3 + x^{3+3} + x^{3+3+3} + \dots)}_{formally\ sums\ to\ \frac{1}{1-x^3}} \cdots$$

*Thus we obtained the generating function for* $p(n)$

$$f(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots = \prod_{i\geq 0}\left(\frac{1}{1-x^i}\right) \tag{3}$$

# Representation of finite groups

**Definition 0.2.** *Let $G$ be a finite group (we shall work mostly with the symmetric group $S_n$). A **Representation** of $G$ is a homomorphism*

$$\phi : \quad G \to GL(V) \tag{4}$$

*Where $V$ is a vector space over some field $\mathbb{K}$ (in our case $\mathbb{K}$ will be the field of complex numbers $\mathbb{C}$).*

**Example 0.3.** *Let $S_n$ be the symmetric group (the group of permutations of the set $\{1, 2, \ldots, n\}$). Let $V = \mathbb{C}^n$ and $e_1, \ldots, e_n$ be the standard basis of $V$ over $\mathbb{C}$. We shall construct a representation of $S_n$ on $V$ as follows:*

1. *Let $\sigma \in S_n$. We set:*

$$\sigma(e_1, \ldots, e_n) \doteq (e_{\sigma(1)}, \ldots, e_{\sigma(n)}) \tag{5}$$

2. *For every element $\sigma \in S_n$ , we define a matrix $\Sigma = \{\Sigma_{ij}\}_{i\,j=1}^{n}$ for the action of $\sigma$ on $V$, described above:*

$$\Sigma_{ij} = \begin{cases} 1 & \textit{for } \sigma(e_i) = e_j \\ 0 & \textit{for } \sigma(e_i) \neq e_j \end{cases} \tag{6}$$

*One can easily see that $det(\Sigma) = sgn(\sigma) = \pm 1$, so $\Sigma \in GL(V)$ and thus we defined a representation of $S_n$ on $V$.*

**Definition 0.3.** *We say that a subspace $W \subset V$ is invariant under the $G$, or $G$-invariant if*

$$\forall w \in W \quad , \quad \phi(g)(w) \in W \quad , \quad \forall g \in G \tag{7}$$

**Example 0.4.**

1. *$W = V$ and $W = 0$ , the trivial subspaces*
2. *Let us consider the representation from Example 0.3 above (for $n \geq 2$). Let*

$$W \doteq span_{\mathbb{C}}\{e_1 + e_2 + \cdots + e_n\} \tag{8}$$

*We have that $\dim_{\mathbb{C}} W = 1$ and, obviously $W$ is $G$-invariant. Moreover, since*

$$\forall g \in G \quad \phi(g)(e_1 + e_2 + \cdots + e_n) = e_1 + e_2 + \cdots + e_n \tag{9}$$

*$G$ acts trivially on $W$, thus the matrix for every element $g \in G$ is $\mathbb{I}_n$- the identity matrix.*

**Definition 0.4.** *A representation $V$ of $G$ is said to be:*

1. ***Reducible** , if $V$ contains a proper (non-trivial) $G$-invariant subspace $W \subset V$*
2. ***Irreducible** , if $V$ does not contain any proper (non-trivial) $G$-invariant subspaces.*
3. ***Completely reducible** , if $V$ is Reducible and in addition*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k \quad , \tag{10}$$

*where all the subspaces $W_i$ , $i = 1, \ldots k$ are Irreducible and $G$-invariant.*

In other words, if $W$ is a reducible representation of $G$, than we can write every element of $G$ as

$$\begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix} \tag{11}$$

Here the first block-row stands for the action of $G$ on $W$ and the second stands for the action of $G$ on $W^c$ - the complement of $W$ in $V$.

**Theorem 0.1.** ***(Maschke)***
*Every reducible representation of a finite group $G$ (with $\dim V \geq 2$ ) is completely reducible, i.e.*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k \quad , \quad W_i \, - \, \textit{irreducible} \tag{12}$$

In other words, for every element $g \in G$, $\phi(g)$ is a block diagonal matrix of the form:

$$\begin{pmatrix} A_1(g) & 0 & \cdots & 0 \\ 0 & A_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k(g) \end{pmatrix} \tag{13}$$

**Definition 0.5.** *Let $\phi : G \to GL(V)$ be a representation of the finite group $G$. Then the* ***Character*** *$\chi_\phi$ of $\phi$ is :*

$$\chi_\phi(g) \mathrel{\hat=} \mathfrak{tr}(\phi(g)) \tag{14}$$

There could be many isomorphic representation of $G$ on the same vector space $V$ , but $\chi$ will be the same for all of them, since we have:

1. If $\phi : G \to GL(V)$ and $\psi : G \to GL(V)$ , then there will exist an element $T \in GL(V)$, such that
$$\forall g \in G \quad \psi(g) = T\phi(g)T^{-1} \tag{15}$$
2. We have
$$\mathfrak{tr}(\psi(g)) = \mathfrak{tr}(T\phi(g)T^{-1}) = \mathfrak{tr}(\phi(g)) \; , \; \textit{as} \tag{16}$$

$$\mathfrak{tr}(AB) = \mathfrak{tr}(BA) \tag{17}$$

**Example 0.5.** *For the representation of $S_n$, defined in Example 0.3 , it is easy to see, that*

$$\chi_\phi(\sigma) = \textit{number of "ones" in on the diagonal of } \phi(g) = \textit{number of fixed points of } \sigma.$$

**Proposition 0.2.** *The character of a representation has the following properties:*

*1.* $\chi(1_G) = \dim(V)$
*2. If $H$ is a conjugacy class in $G$, we have*
$$\forall f, h \in H \quad \chi(f) = \chi(h)$$
*3. Let $V$ and $W$ are representations of $G$, then*
$$V \simeq W \quad \Leftrightarrow \quad \chi_V(g) = \chi_W(g) \; , \; \forall g \in G$$

We end this introductory section stating a theorem that we shall use later.

**Theorem 0.3.** *The number of irreducible complex representations of a finite group $G$ is equal to the number of conjugacy classes in $G$. Moreover, if $V_1, \ldots V_n$ are all the irreducible representations of $G$, we have*

$$\sum_{i=1}^{s} (\dim V_i)^2 = |G| \tag{18}$$

## Partitions and Young tableaux

**Definition 0.6.** *Let $n \in \mathbb{N}$. A* ***Partition*** *of $n$ is*

$$\lambda \vdash n \quad \lambda = (\lambda_1, \ldots, \lambda_l)\ ,\ \lambda_i \in \mathbb{N} \tag{19}$$

*such that*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \quad , \quad \sum_{i=1}^{l} \lambda_i = n \tag{20}$$

**Definition 0.7.** *The* ***Shape*** *of $\lambda$ is a diagram, formed by boxes, aligned to the left, such that there are $l$ rows and the $i$-th row contains exactly $\lambda_i$ boxes.*

**Example 0.6.** *Let $\lambda = (3, 3, 2, 1)$. In this case we have that $\lambda \vdash 9$ and the Shape of $\lambda$ is:*

$$\ydiagram{3,3,2,1}$$

Given a shape $\lambda$, we define the conjugate shape $\lambda'$ simply as a mirror reflection of $\lambda$ by the main diagonal. The conjugate of the tableau above will be the following

$$\lambda = \ydiagram{3,3,2,1} \qquad \mapsto \qquad \lambda' = \ydiagram{4,3,2} \tag{21}$$

**Definition 0.8.** *Let $\lambda$ be a shape $\lambda \vdash n$. A* ***Young tableau*** *is a numbering of $\lambda$ with some positive integers,such that*

1. *In each row the numbers are weakly increasing*

2. *In each column the numbers are strictly increasing*

*A tableau is called* ***Standard****, if its shape $\lambda \vdash n$ is numbered with the integers $\{1, 2, \ldots n\}$.*

We would need the following very useful construction:

**<u>Row bumping algorithm</u>**

Given a Young tableau $\lambda \vdash n$, we can "insert" another number $p \in \mathbb{N}$ in it by following the steps below:

1. Look at the first row of $\lambda$. If $p$ is at least as large as all the numbers in that row, simply add a new box at the end of it and write $p$ in it.

2. If there is a number, which is strictly bigger that $p$, find the left-most entry of the row, that is strictly bigger than $p$. Remove that number from its box and put $p$ inside it.

3. Take the "bumped value" and repeat the steps for the second row

**Example 0.7.** *We shall bump the value 2 in the Young tableau below. Every time we bump an entry, we shall inscribe a black square in the box, just to demonstrate the step, and later insert the "queued" value inside:*

$$\lambda = \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 \\ 5 & 6 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 2 & \blacksquare \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 \\ 5 & 6 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 2 & 2 \\ 2 & 3 & \blacksquare & 5 \\ 4 & 4 & 6 \\ 5 & 6 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & \blacksquare \\ 5 & 6 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 \\ 5 & 6 & 6 \end{ytableau} \tag{22}$$

**Proposition 0.4.** *The Row Bumping preserves the property that the rows will be increasing and the columns-strictly increasing, thus the result is again a Young tableau*

*Proof.* The proposition follows easily from the algorithm steps. $\square$

**Remark 0.1.** *The Row Bumping algorithm is reversible if we know the exact box where we inserted the additional value (in our case this is the additional value 2). In our example above, if we know that the value 2 has been bumped in the last box of the first row of $\lambda$, we can easily execute the algorithm backwards and obtain the initial Young tableau $\lambda$.*

Our next aim is to prove the Robinson-Schensted(-Knuth) Correspondence. In fact Knuth generalized Schensted's generalization of Robinson's correspondence. We shall use (and therefore construct) only the Robinson-Schensted correspondence.

**Theorem 0.5.** *There is a one-to-one correspondence between the elements of $S_n$ and the pairs of standard Young tableaux with $n$ boxes of the same shape.*

**Remark 0.2.** *We shall see from the construction of the correspondence that all possible shapes $\lambda \vdash n$ will be obtained.*

*Proof.* Expectedly, the proof is constructive. We shall start with a permutation $\sigma \in S_n$ and construct a pair of standard Young tableaux that corresponds to it. Let $\sigma \in S_n$. Then $\sigma$ can be written as:

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix} \tag{23}$$

We construct a sequence of pairs of standard Young tableaux as follows:

1. Let $(P_0, Q_0)$ be a pair of empty Young tableaux
2. Let $P_1$ be consisted only of 1 box with value $\sigma(1)$ and $Q_1$ be consisted of 1 box with value 1
3. We start to bump $\sigma(i)$ inside $P(1)$. At each step, we obtain a new $P_i$. Once we bump $\sigma(i)$ in $P_{i-1}$, we add a new box in $Q_{i-1}$ at exactly the same place, where a new box has been added to $P_{i-1}$ and put inside the value i.

$Q_i$ are called the recording tableaux. Why would we need them? Recall that the row bumping is reversible if we know which was the box that we bumped initially. Q will tell us the order of bumping, so the described algorithm will be reversible as well. Thus we have that the algorithm is one-to-one.

□

**Example 0.8.** *Let $\sigma \in S_8$ be the following permutation*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix} \tag{24}$$

*Then following the algorithm above (I will skip the different steps, since they are easy but rather long), we obtain that the pair of Young tableaux, corresponding to $\sigma$ is*

$$P = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 6 \\ \hline 3 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} \quad , \quad Q = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 8 \\ \hline 3 & 4 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

**Corollary 0.6.** *If $\sigma \mapsto (P, Q)$, then $\sigma^{-1} \mapsto (Q, P)$.*

**Remark 0.3.** *We thus also showed the formula*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n! \tag{25}$$

*Where the sum is over all partitions of $n$, and $f^\lambda$ = number of standard Young tableaux of shape $\lambda$.*

We shall now introduce a geometric construction of the Robinson-Schensted correspondence which will prove useful in the following sections.

**Viennot's Geometric construction**

Let $\sigma \in S_n$ be the permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix} \tag{26}$$

Lets plot the points $(i, \sigma(i))$ in the plane $\mathbb{R}^2$. The result will be $n$ different point in the first quadrant. We construct the, so called, *shadow diagram* of $\sigma$ in the following way:

1. Take any of the points $(i, \sigma(i))$. Draw two rays, starting at this point, one parallel to the $x$-axis and pointing to the right and one parallel to the $y$-axis pointing upwards.

2. Repeat the same procedure for all the points

3. Check all the points where the shadow lines of two distinct points cross each other and cut them there.

Thus we obtain a family of broken lines, called the *shadow lines* of $\sigma$, consisting of line segments and exactly one vertical and one horizontal rays for each of them.

Next, from our construction it follows that every ray will lie on a line of the form $x = k$ , $k \in \mathbb{N}$ or $y = k$ , $k \in \mathbb{N}$. Take those numbers $k$ for all the horizontal rays in increasing order and put them in boxes, thus forming the first row of a Young tableau. Proceed the same way with the vertical rays and form the first row of another Young tableau. Call the first tableau $P$ and the second $Q$. We shall insert more rows in the next steps of the algorithm.

The second step is the following:

1. The shadow lines of $\sigma$ are broken lines, as described above. Consider all the points, where the lines break (the angle is always $\pi/2$ by construction). All the points $(i, \sigma(i))$ are breaking points, but we already dealt with them, so delete them. Consider all the points that rest. Mark them and draw their shadow diagram.

2. Repeat the previous step until there are no further angles, which we have not considered earlier.

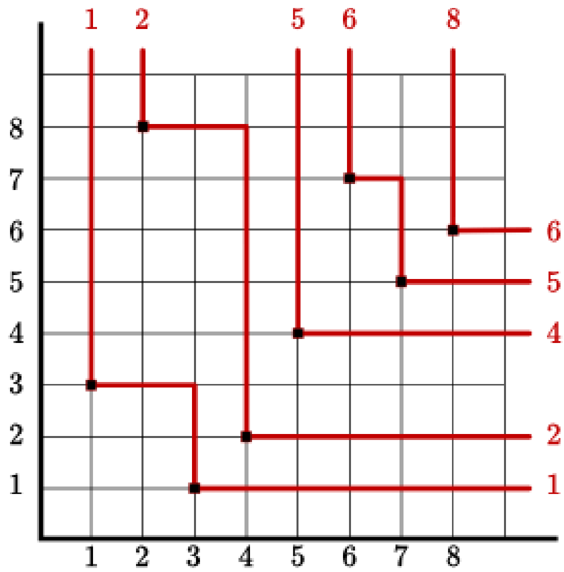
At each step, we shall obtain a new row for $P$ and $Q$ and after a finite number of steps, we shall obtain the whole tableaux $P$ and $Q$.

**Example 0.9.** *In order to clarify and illustrate this algorithm, we shall perform this construction for our old sport*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix} \tag{27}$$

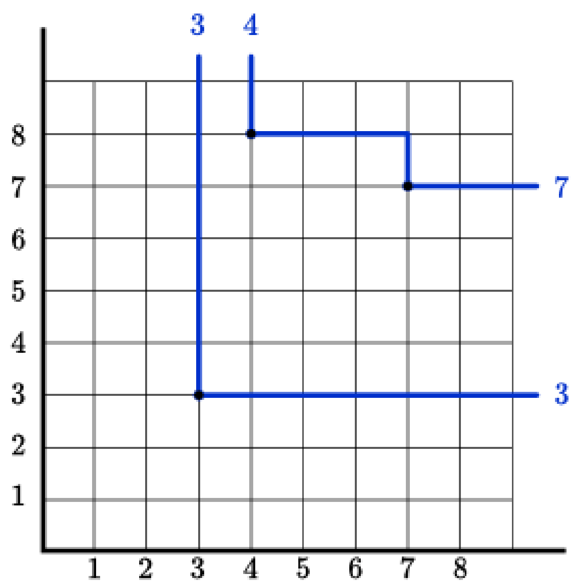
*(The pictures above were taken from wikipedia.org)*

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&5&6&8\\\hline\end{array}$$




$$\begin{array}{|c|c|c|c|c|}\hline 1&2&4&5&6\\\hline\end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&5&6&8\\\hline 3&4\\\cline{1-2}\end{array}$$




$$\begin{array}{|c|c|c|c|c|}\hline 1&2&4&5&6\\\hline 3&7\\\cline{1-2}\end{array}$$

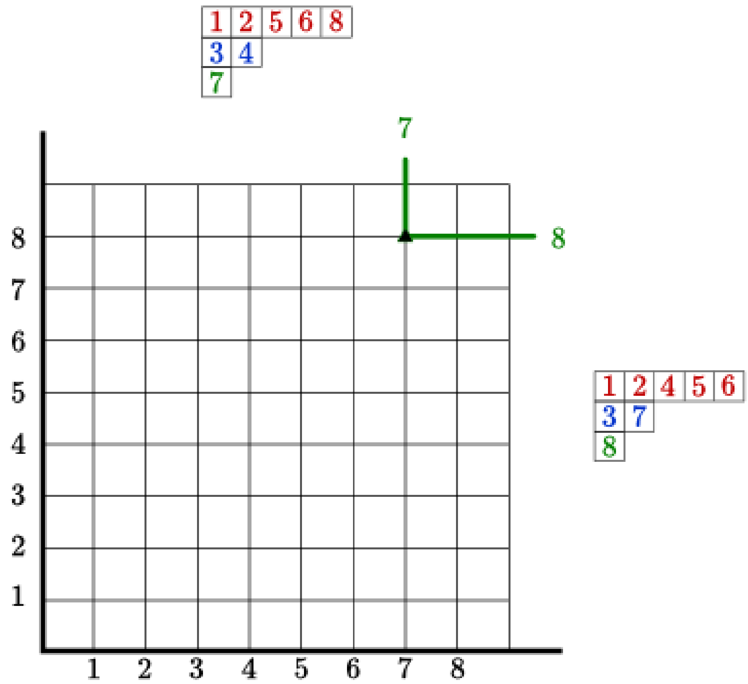
*So the two matrices we obtain are:*

$$P = \begin{array}{|c|c|c|c|c|}\hline 1&2&4&5&6\\\hline 3&7\\\cline{1-2} 8\\\cline{1-1}\end{array} \quad , \quad Q = \begin{array}{|c|c|c|c|c|}\hline 1&2&5&6&8\\\hline 3&4\\\cline{1-2} 7\\\cline{1-1}\end{array}$$

*Now just notice that those are the same matrices as in Example 0.8 , so we obtained the matrices that correspond to $\sigma$ under the Robinson-Schensted correspondence.*

We shall conclude this section with the formulation of the result we just observed.

**Proposition 0.7.** *For every permutation $\sigma \in S_n$, Viennot's geometric construction for $\sigma$ produces the same pair of standard Young tableaux as the Robinson-Schensted correspondence does.*




The proof of the proposition is purely combinatorial and will not be needed for our further considerations, thus will be skipped.

# The ring of symmetric functions

**Definition 0.9.** *We shall denote by $\Lambda_n$ the set of symmetric polynomials of $n$ variables with complex coefficients, i.e.*

$$\Lambda_n \circeq \{p \in \mathbb{C}[x_1, \dots, x_n] \mid p(\sigma(x_1), \dots, \sigma(x_n)) = p(x_1, \dots, x_n) \ , \ \forall \sigma \in S_n\} \tag{28}$$

$\Lambda_n$ has the structure of a graded ring over $\mathbb{C}$, namely

$$\Lambda_n = \bigoplus_k \Lambda_n^k \tag{29}$$

where $\Lambda_n^k$ is the set of homogeneous symmetric polynomials $p$ with $\deg(p) = k$, together with the zero polynomial. Each $\Lambda_n^k$ is an abelian group (additive) and in addition we have

$$\Lambda_n^i \Lambda_n^j \subset \Lambda_n^{i+j} \tag{30}$$

(The last two equations being essentially the definition of a graded ring).

In the following, by $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we shall denote multi-indexes. Also, for $\alpha$ - a multi-index, we shall denote by $x^\alpha$ the monomial

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n} \tag{31}$$

Let $\lambda \vdash k$ for some $k \in \mathbb{N}$ , such that $l(\lambda) \leq n$, i.e $\lambda = (\lambda_1, \dots, \lambda_n)$ , $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , $\sum_i = k$. Here each $\lambda_i$ is taken with some multiplicity and some of them can be actually zero, provided that the other have such a multiplicity that the sum is still equal to $k$. Let

$$m_\lambda \circeq \sum_{\sigma(\lambda)} x^\lambda \tag{32}$$

where the sum is taken over all distinct permutations of $\lambda_1, \dots, \lambda_n$), i.e. for $\sigma \in S_n$ , $\sigma(\lambda) = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$ (this might not be a partition any more). Also in the sum we regards $\lambda$ as a multi-index.

Clearly $m_\lambda$ is a symmetric polynomial for every $\lambda$. Moreover $m_\lambda$ forms a $\mathbb{C}$-basis of $\Lambda_n$, as $\lambda$ runs through all partitions with length $l(\lambda) \leq n$ (Remember that $l(\lambda) \leq n$ is only a restriction on the number of distinct non-zero integers in $\lambda$ but not on $|\lambda|$, so we can get $\deg(m_\lambda)$ as big as we want). Also the $m_\lambda$, such that $|\lambda| = k$ give a $\mathbb{C}$-basis of $\Lambda_n^k$.

Our next aim is to build the ring of symmetric functions over $\mathbb{C}$. First, for $m \geq n$ we introduce the homomorphisms:

$$\rho_{m,n} : \Lambda_m \to \Lambda_n \tag{33}$$

which send all the variables $(x_{n+1}, \dots, x_m)$ to 0. The effect of $\rho_{m,n}$ on the basis $m_\lambda$ is easily described as well:

1. if $l(\lambda) \leq n$, $\rho_{m,n}$ maps $m_\lambda(x_1, \dots, x_m)$ to $m_\lambda(x_1, \dots, x_n)$
2. if $l(\lambda) > n$, $\rho_{m,n}$ maps $m_\lambda(x_1, \dots, x_m)$ to 0

It follows that $\rho_{m,n}$ is surjective. On the restriction to $\Lambda_n^k$ we have homomorphisms:

$$\rho_{m,n}^k : \quad \Lambda_m^k \to \Lambda_n^k \tag{34}$$

for all $k \geq 0$ , $m \geq n$, which are surjective. Moreover, it is easy to see (since we are looking only at symmetric polynomials), that for $m \geq n \geq k$, $\rho_{m,n}^k$ is actually an isomorphism.

**Definition 0.10.** *Let $\{A_i\}_{i \in \mathbb{N}}$ be a family of groups. Let $\{\rho_{j,i}\}_{0 \leq i \leq j}$ be a family of group-homomorphisms, such that*

1. $\rho_{j,i} : \quad A_j \to A_i \quad , \quad j \geq i$
2. $\rho_{i,i} = \mathbb{I}_{A_i}$
3. $\rho_{i,k} = \rho_{i,j} \circ \rho_{j,k}$ *, for* $\ i \leq j \leq k$

*Then the Projective Limit of $\{A_i\}_{i\in\mathbb{N}}$, with respect to the homomorphisms $\{\rho_{j,i}\}_{0\leq i\leq j}$ is the group, defined by*

$$\varprojlim_i A_i \doteq \{a = (a_1, a_2, \dots) \in \prod_i A_i \quad | \quad p_{j,i}(a_j) = a_i\} \tag{35}$$

*The projective limit is a special case of the inverse limit, namely when the homomorphisms $\rho_{i,j}$ are epimorphisms (as is in our case).*

Next we form the projective limits of the homogeneous components (which are vector spaces over $\mathbb{C}$) $\Lambda_n^k$ , with respect to the homomorphisms $\rho_{m,n}^k$. Let

$$\Lambda^k = \varprojlim_n \Lambda_n^k \tag{36}$$

Thus, by definition, an element of $f \in \Lambda^k$ will be a sequence $f = (f_n)_{n\geq 0}$ , such that $f_n(x_1, \dots, x_n)$ is a symmetric homogeneous polynomial with $\deg(f) = k$ and for $m \geq n$ , $f_m(x_1, \dots, x_n, 0, \dots, 0) = f_n(x_1, \dots, x_n)$. As we saw, $\rho_{m,n}^k$ is an isomorphism for $m \geq n \geq k$, so the projection

$$\rho_n^k : \quad \Lambda^k \to \Lambda_n \tag{37}$$

which sends $f$ in $f_n$, is an isomorphism for $n \geq k$. Thus we obtain that $\Lambda^k$ has a $\mathbb{C}$-basis consisting of the monomial symmetric functions $m_\lambda$ (for all partitions $\lambda \vdash k$), defined by:

$$\rho_n^k(m_\lambda) = m_\lambda(x_1, \dots, x_n) \tag{38}$$

for all $n \geq k$. Hence $\Lambda^k$ is a linear space with $\dim(\Lambda^k) = p(k)$, where $p(k)$ is the number of partitions of $k$.

Now let

$$\Lambda \doteq \bigoplus_{k\geq 0} \Lambda^k \tag{39}$$

so that $\Lambda$ is a vector space, generated (as a free $\mathbb{C}$-module) by the symmetric monomials $m_\lambda$ for all partitions $\lambda$ (we do not imply any restrictions on $\lambda$, i.e. $l(\lambda)$ and $|\lambda|$ can be arbitrarily big).
We have the surjective homomorphisms:

$$\rho_n \doteq \bigoplus_{k\geq 0} \rho_n^k : \quad \Lambda \to \Lambda_n \tag{40}$$

for each $n \geq 0$ and in addition $\rho$ is an isomorphism in degrees $k \leq n$.

**Remark 0.4.** *By construction $\Lambda$ has the structure of a graded ring, called the ring of symmetric functions over $\mathbb{C}$. The elements of $\Lambda$ can be explicitly described as:*

$$\Lambda = \{(f_1, f_2, f_3, \dots) \,|\, f_i \in \Lambda_i \ , \ \rho_{m,m-1}(f_m) = f_{m-1} \ , \ deg(f_i) < \infty\} \tag{41}$$

*With the additional condition that, if we regard the elements of $\Lambda$ as formal power series, by construction we would have $\forall f \in \Lambda$ , $\deg(f) < \infty$. The boundedness is essential.*

**Remark 0.5.** *I would like to give a bit more details concerning the boundedness of the degree of the elements in $\Lambda$. Namely, as per our construction, we would have that*

$$\Lambda = \varprojlim_n \Lambda_n \tag{42}$$

*in the category of graded rings. However, if we just take the rings $\Lambda_n$ and form their projective limit in the category if rings, with respect to the homomorphisms $\rho_{m,n}$ we shall obtain a ring, which is different of $\Lambda$. For example the infinite product $\prod_{i=1}^{\infty}(1 + x_i)$ would belong to this projective limit, but it does not belong to $\Lambda$.*

**Schur Polynomials**

We shall now introduce the Schur polynomials.

Let $\lambda \vdash m$ be a partition with at most $m$ rows. Let $T$ be a *numbering* of $\lambda$ with the numbers $(1, \dots m)$, i.e.

$$T = (1^{t_1}, 2^{t_2}, \dots, m^{t_m})$$

where $t_i$ is the multiplicity of $i$ in $T$. For each numbering $T$ of $\lambda$, we shall define a monomial by

$$x^T \doteq x_1^{t_1} x_2^{t_2} \dots x_m^{t_m} \tag{43}$$

**Example 0.10.** *Let*

$$\lambda = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \cline{1-4} \ & \ & \ & \ \\ \cline{1-4} \ & \ \\ \cline{1-2}\end{array} \qquad T = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 3 & 5 \\ \hline 2 & 3 & 5 & 5 \\ \cline{1-4} 4 & 4 & 6 & 6 \\ \cline{1-4} 5 & 6 \\ \cline{1-2}\end{array}$$

*Then*

$$x^T = x_1 \ x_2^3 \ x_3^3 \ x_4^2 \ x_5^4 \ x_6^3$$

**Definition 0.11.** *The **Schur polynomial** of the shape $\lambda$ is*

$$s_\lambda(x_1, \dots, x_m) \doteq \sum_T x^T \tag{44}$$

*Where the sum goes through all the numberings $T$ of $\lambda$, which give a Young tableau.*

**Example 0.11.** *Two special cases are worth mentioning:*

1. *Let $\lambda$ has only one row with $n$ boxes ($\lambda = (n)$). Then $s_\lambda(x_1, \dots x_m)$ will be the $n$-th complete symmetric homogeneous polynomial, i.e. $h_n(x_1, \dots, x_m) = \sum_{1 \le i_1 \le \dots \le i_n \le m} x_{i_1} \ \dots \ x_{i_n}$ . For example let $\lambda = \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$ and $m = 2$. Then $s_\lambda(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$*

2. *Let $\lambda$ has only one column with $n$ boxes ($\lambda = (1^n)$) (here we can see why the number of rows should not exceed the number of variables -m). Then $s_\lambda(x_1, \dots x_m)$ will be the $n$-th elementary symmetric polynomial $e_n(x_1, \dots, x_m) = \sum_{1 < i_1 < \dots < i_n < m} x_{i_1} \ \dots \ x_{i_n}$ For example let $\lambda = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$ and $m = 2$. Then $s_\lambda(x_1, x_2) = x_1 x_2$ .*

**Proposition 0.8.** *The Schur polynomials are symmetric. Moreover, the set of the Schur polynomials $s_\lambda$ for all $\lambda$, such that $l(\lambda) \le m$, forms a $\mathbb{C}$-basis of $\Lambda_n$.*

*Proof.* The proof of the fact that the Schur polynomials are symmetric is easy and swiftly follows from the definition. The proof that they span $\Lambda_n$ will be skipped here. □

We would like to see what elements in $\Lambda$ will correspond to the Schur polynomials. First we observe that if $l(\lambda) \le N$

$$\rho_{N+1,N}(s_\lambda)(x_1, \dots, x_N, x_{N+1}) = s_\lambda(x_1, \dots, x_N) \tag{45}$$

In addition,

$$\rho_{l(\lambda),l(\lambda)-1}(s_\lambda)(x_1, \dots, x_{l(\lambda)}) = 0 \tag{46}$$

Therefore, for every fixed $\lambda$ the sequence of polynomials $s_\lambda(x_1, \dots, x_N)$ with varying number of variables $N \ge l(\lambda)$, completed by zeros for $N < l(\lambda)$, will define an element of $\Lambda$. We shall call this element *The Schur symmetric function* $s_\lambda$. By definition $s_\emptyset = 1$.

**Theorem 0.9.** *The Schur functions $s_\lambda$, with $\lambda$ ranging over the set of all partitions (shapes), form a linear basis of $\Lambda$ over $\mathbb{C}$.*

Now suppose that we have two copies of the algebra $\Lambda$ or, in other words, two sets of variables $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. We shall consider functions of the form $s_\lambda(x)\, s_\mu(y)$, which will be symmetric functions in variables $x$ and $y$ separately, but not jointly. Formally such functions can be viewed as elements of the the tensor product $\Lambda \bigotimes_{\mathbb{C}} \Lambda$.

More generally, we can consider an infinite sum

$$\sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) \tag{47}$$

as an infinite series symmetric in variables $x_1, x_2, \ldots$ and in variables $y_1, y_2, \ldots$. The following theorem will give us a formula for this sum in the language of generating functions.

**Theorem 0.10.** *(The Cauchy identity) In terms of generating functions, we have the following identity:*

$$\sum_{\lambda} s_{\lambda}(x_1, x_2, \ldots)s_{\lambda}(y_1, y_2, \ldots) = \prod_{i,j} \frac{1}{1 - x_i y_j} \quad , \tag{48}$$

*where the sum on the left goes over the set of all partitions (shapes).*

The proof will be skipped here.

We conclude this section introducing the term *specialization* , which will be used later.

**Definition 0.12.** *Any algebra homomorphism* $\rho : \quad \Lambda \to \mathbb{C} \quad , \quad f \mapsto f(\rho)$ *(the latter notation will become clear in the what follows), is called a specialization. In other words,* $\rho$ *should satisfy the following properties:*

1. $(f + g)(\rho) = f(\rho) + g(\rho)$
2. $(fg)(\rho) = f(\rho)g(\rho)$
3. $\forall \theta \in \mathbb{C} \;\; , \;\; (\theta f)(\rho) = \theta f(\rho)$

Take any sequence of complex numbers $\{u_i\}_{i \geq 0}$ , satisfying $\sum_{i \geq 0} |u_i| < \infty$. Then the *substitution* map $\Lambda \to \mathbb{C}$ is defined by $x_i \mapsto u_i$. In other words, the image of an element in $\Lambda$ is an infinite sum of monomials of the elements $(u_1, u_2, \ldots)$ , which will be convergent because of the way we selected the complex numbers $\{u_i\}_i \geq 0$.

**Definition 0.13.** *Let* $\rho$ *be a specialization. We shall call* $\rho$ *Schur-positive if for every shape* $\lambda$*, we have:*

$$s_{\lambda}(\rho) \geq 0 \tag{49}$$

## Probabilistic models from the "KPZ Class", related to what we introduced so far

We are finally ready to consider our main object of interest. We shall introduced the, so called *Totally Asymmetric Simple Exclusion Process*, abbreviated TASEP. This is one of the fundamental two dimensional growth models in the KPZ Universality class. Consider the interface which is a broken line, consisting of straight pieces with slope $+1$ and $-1$ (as shown in the *Figure 1* below) and suppose a new unit box is added at each local minimum independently after an exponential waiting time.

An alternative formulation of this growth model can be constructed as follows. Project the interface to a straight line and put "particles" at the projection of unit segments of slope $-1$ and "holes" at unit segments of slope $+1$ (See *Figure 1* Right picture). Now each particle independently jumps to the right after an exponential waiting time, with the restriction that jumps to already occupied spots are prohibited.

Figure 1: The TASEP model in the general case

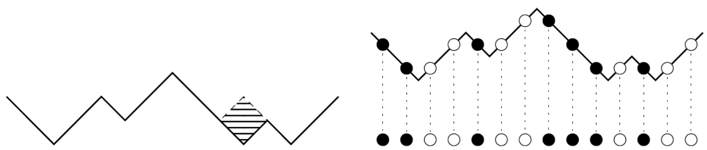
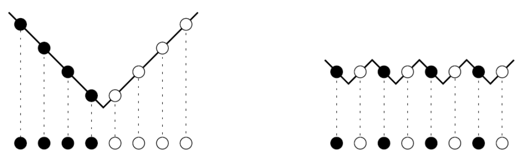


Figure 2: Left: A "wedge" initial interface ; Right: A "flat" initial interface



The two models in *Figure 2* are special cases of the TASEP. For them we shall state the following results:

**Theorem 0.11.** *Suppose that at time* $0$ *the interface* $h(x;t)$ *is a wedge* $h(x,0) = |x|$*, as shown in Figure 2 (left picture). Then for every* $x \in (-1,1)$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{h(t,tx) - c_1(x)t}{c_2(x)t^{\frac{1}{3}}} \geq -s\right) = F_2(s) \tag{50}$$

*where* $c_1(x), c_2(x)$ *are certain (explicit) functions of* $x$*.*

**Theorem 0.12.** *Suppose that at time* $0$ *the interface* $h(x;t)$ *is flat, as shown in Figure 2 (right picture). Then for every* $x \in \mathbb{R}$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{h(t,x) - c_3 t}{c_4 t^{\frac{1}{3}}} \geq -s\right) = F_1(s) \tag{51}$$

*where* $c_1, c_2$ *are certain (explicit) positive constants.*

Here $F_1(s)$ and $F_2(s)$ are the Tracy-Widom distributions. They are the limiting distributions for the largest eigenvalues in Gaussian Orthogonal Ensemble and Gaussian Unitary Ensemble of random matrices.

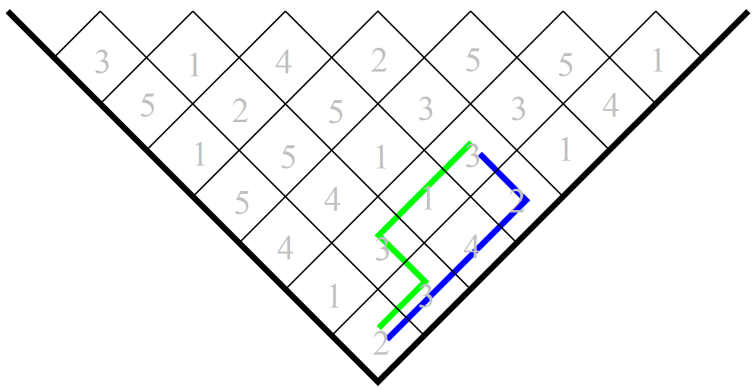
These two theorems give the conjectural answer for the whole " KPZ universality class" of two dimensional random growth models. Let us remark here that the fluctuation is of order $t^{\frac{1}{3}}$, and not $t^{\frac{1}{2}}$, as in the classical central limit theorem. Also, the distribution on the right side may vary from model to model. Conjecturally, the only two generic subclasses are the ones we have seen.

Let us concentrate on the wedge initial condition. In this case we can reformulate the model once again. Consider the first quadrant of $\mathbb{R}^2$ with the integer lattice $\mathbb{Z}^2$ in it. In each box $(i,j)$ write a random "waiting time" $w_{(i,j)}$. Once our random interface (of wedge type) reaches the box $(i,j)$, it takes time $w_{(i,j)}$ for it to percolate and absorb it. Now the whole quadrant is filled with i.i.d variables. It is not hard to reconstruct our initial TASEP model from the one we just constructed. Let $T(i,j)$ be the time at which our growing interface will absorb the box $(i,j)$. It is not hard to see that

$$T(i,j) = \max_{(1,1)=b[1]\to b[2]\to\cdots\to b[i+j-1]=(i,j)} \sum_{k=1}^{i+j-1} w_b[k] \quad , \tag{52}$$

where the sum is taken over all the directed (leading away from the origin) paths, joining $(1,1)$ and $(i,j)$ (see *Figure 3*). In other words, (52) gives the worst case scenario for the percolation from $(1,1)$ in $(i,j)$.

Figure 3: The Last Passage Percolation




We shall focus on the LPP and show explicitly its connection to the notions we introduced in the previous sections. Let us consider a specific limit case of it, when $w_{(i,j)}$ takes only two values 0 and 1, and the probability of 1 is very small. We can see this limit as follows:
Consider the homogeneous density 1 the Poisson Point Process in the first quadrant. Let $L(\theta)$ be the maximal number of points one can collect along a North-East path from $(0,0)$ to $(\theta,\theta)$. We consider only directed paths, as for the general LPP, so we cannot go back.

Just to recall that the Poisson Point Process is characterized by the following two properties

1. The numbers of isolated points falling within two regions $A$ and $B$ are independent random variables if A and B do not intersect each other

2. The probability distribution of the number $X$ of isolated points falling within any region $A$ is a Poisson distribution with parameter $|A|$, namely

$$\mathbb{P}(X=k) = \frac{|A|^k e^{-|A|}}{k!} \tag{53}$$

On *Figure 4* we can easily see that what we constructed above may be regarded as a specific limit of the LPP. The following theorem has been proved for this model.

**Theorem 0.13.**

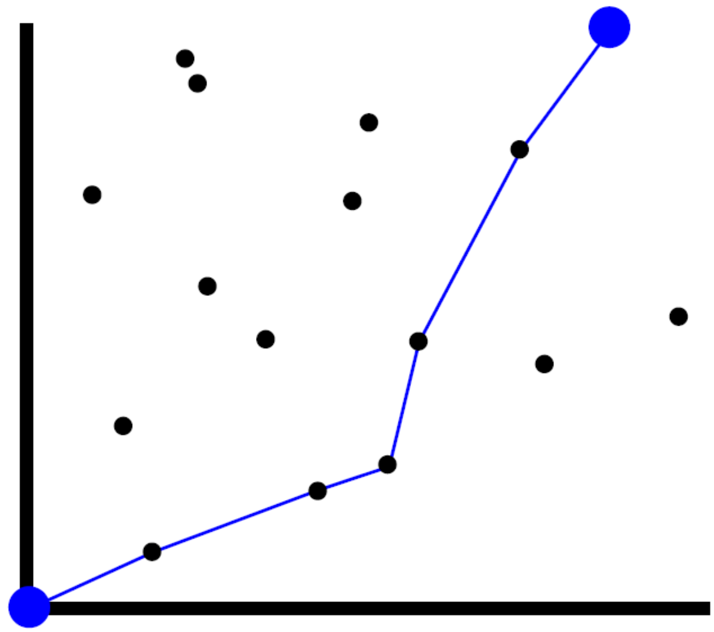
$$\lim_{\theta\to\infty} \mathbb{P}\left(\frac{L(\theta)-2\theta}{\theta^{\frac{1}{3}}} \le s\right) = F_2(s) \tag{54}$$

*where again $F_2(s)$ is the Tracy-Widom distribution.*

Indeed this model is in the KPZ Universality Class.

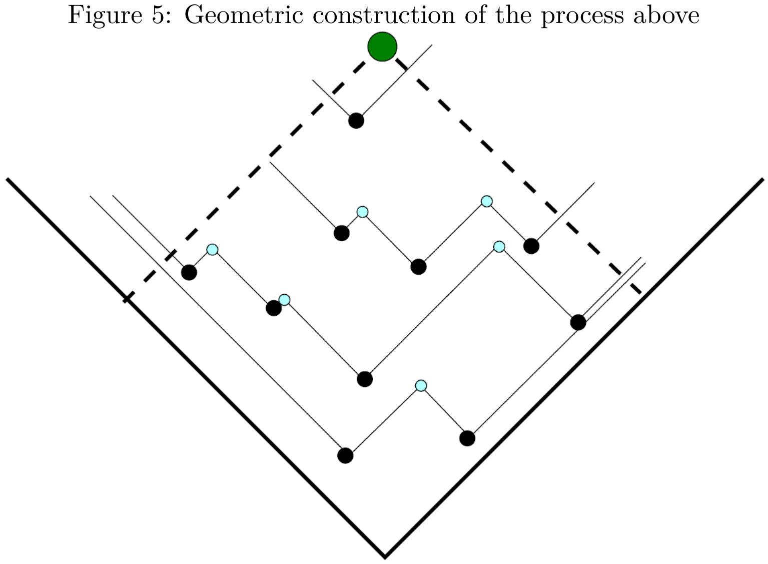
We can reconsider the model above in the following manner. First, lets take again the Poisson Point Process in the first quadrant, but this time, rotate the coordinate system by $\frac{\pi}{4}$ to the left (the reason why we rotate it

Figure 4: The limit of the LPP we described above



will be revealed later). Now for each point in the process, draw two rays parallel to the coordinate axes. Extend each ray till it meet another ray. Thus we get a collection of broken lines (each of them will have exactly one vertical and one horizontal rays) (See *Figure 5*). Note now that $L(\theta)$ is equal to the number of broken lines, separating the origin $(0,0)$ and the point $(\theta,\theta)$.

Figure 5: Geometric construction of the process above



It turns out beneficial to iterate the process. We erase all the points from the Poisson Point Process, but keep the points where the rays were intersecting. We repeat the same construction as above for those points, thus generating new points of intersection. We continue like that until no points are left in the square with vertices $(0,0),(0,\theta),(\theta,0)$ and $(\theta,\theta)$. Compute the number of broken lines separating $(0,0)$ and $(\theta,\theta)$ at each step and record those numbers to form a shape (a partition) $\lambda=(\lambda_1(\theta),\ldots,\lambda_l(\theta))$. We can observe a couple of things here. First, we have $\lambda_1(\theta)=L(\theta)$ and $|\lambda(\theta)|$ equals the number of the points in the initial Poisson Point Process.

In order to give a formula for the distribution of $\lambda(\theta)$, we first need to introduce a probabilistic measure on the set of shapes. We start with the following

**Definition 0.14.** *Let $n\in\mathbb{N}$. The* ***Plancherel measure*** *is a measure on the set of all partitions (shapes) $\lambda\vdash n$,* *defined by*

$$\mu_n(\lambda) = \frac{(f^\lambda)^2}{n!} \tag{55}$$

*where $f^\lambda$ is the number of standard Young tableaux of shape $\lambda$.*

**Remark 0.6.** *Originally the Plancherel measure is defined on the set of all irreducible representations of a finite group $G$ (we consider the symmetric group $S_n$) by*

$$\mu_n(\phi) = \frac{(\dim \phi)^2}{n!}$$

*However, for the symmetric group there is a one-to-one correspondence between the set of its irreducible representations and the set of all shapes $\lambda \vdash n$. Moreover, one can show that for each irreducible representation $\phi$ of $S_n$, $\dim \phi$ is equal to the number of standard Young tableaux of shape $\lambda \vdash n$.*

Now, out of the Plancherel measures for all $n \in \mathbb{N}$ we shall construct a probabilistic measure on the set of all shapes (we shall denote this set by $\mathbb{Y}$). The process we shall use is called *Poissonization*.

First, we observe that the Robinson-Schensted correspondence gives us

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n! \tag{56}$$

so if we sum the the measures of all the shapes $\lambda \vdash n$ for any fixed $n$, we shall get 1.

**Definition 0.15.** *For $\theta > 0$, the* ***Poissonized Plancherel measure*** *$M^\theta$ is a probabilistic measure on the set of all shapes, defined by*

$$M^\theta(\lambda) = e^{-\theta} \sum_{n \in \mathbb{N}} \frac{\theta^n}{n!} \mu_n(\lambda) = e^{-\theta} \theta^{|\lambda|} \left( \frac{f^\lambda}{|\lambda|!} \right)^2 \tag{57}$$

Now we are ready to formulate our theorem:

**Theorem 0.14.** *The distribution of $\lambda(\theta)$ is given by the Poissonized Plancherel measure*

$$\mathbb{P}(\lambda(\theta) = \mu) = e^{-\theta^2} \left( \frac{\theta^{|\mu|} f^\mu}{|\mu|!} \right)^2 \quad , \quad \mu \in \mathbb{Y} \tag{58}$$

We shall not prove the theorem, but only mention that the proof is based on the properties of the Poisson Point Process and the Robinson-Schensted correspondence and is not particularly hard (unlike all the other theorems concerning distributions for some models, that we stated here).

**Remark:** I spent a lot of time searching for the proper papers and books, thus gathering a good amount of them. So along with the ones I actually read or at least browsed, I will also include the rest of those I found, in case anyone is interested.

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