Gaussian noise stability and Gaussian isoperimetric inequality

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In these notes, we give an introduction to the Gaussian isoperimetric inequality and a generalization in form of the Gaussian noise stability. The Gaussian isoperimetric inequality arises naturally as an infinite dimensional version of the Euclidean isoperimetric inequality where the optimizers for a fixed Gaussian measure are half-spaces instead of balls.

The Gaussian noise stability of a measurable set is the probability that two standard Gaussian vectors with correlation ρ both belong to this set. Fixing the Gaussian measure of the set half-spaces maximize this probability which we prove by applying techniques from stochastic calculus. This statement contains the Gaussian isoperimetric inequality as special case in the limit $\rho \to 1$.

Contents

1. Introduction

One of the oldest mathematical problems is the isoperimetric inequality in two dimensions [8, 12]. An isoperimetric inequality connects the volume of a set with its surface area. The isoperimetric inequality in $\mathbb{R}^n$ asserts that for every compact subset $A \subset \mathbb{R}^n$ with smooth boundary ∂A and every ball $B \subset \mathbb{R}^n$ with $\mathrm{vol}_n(A) = \mathrm{vol}_n(B)$ we have the inequality

$$\mathrm{vol}_{n-1}(\partial A) \geq \mathrm{vol}_{n-1}(\partial B).$$

An equivalent formulation is given by

$$\mathrm{vol}_n(A_r) \geq \mathrm{vol}_n(B_r)$$

where $M_r := \{x \in X; d(x, y) \leq r \text{ for some } y \in M\}$ is the r-extension of a set $M \subset X$ in a metric space (X, d). (Clearly, we consider $X = \mathbb{R}^n$ with the Euclidean distance.) The equivalence of these two statements can be proved via Minkowski's formula

$$\mathrm{vol}_{n-1}(\partial A) = \liminf_{r \downarrow 0} \frac{1}{r} \left[\mathrm{vol}_n(A_r) - \mathrm{vol}_n(A)\right] \tag{1.1}$$

for a sufficiently smooth boundary ∂A.

Moreover, there is the following isoperimetric inequality on the sphere S^N_ρ in $\mathbb{R}^{N+1}$ with radius ρ. The sphere S^N_ρ is equipped with the geodesic distance as metric and the normalized rotationally invariant measure σ^N_ρ.

Theorem 1.1. *Let $A \subset S_\rho^N$ be a measurable subset and $B \subset S_\rho^N$ a geodesic ball such that $\sigma_\rho^N(A) \geq \sigma_\rho^N(B)$ then*

$$\sigma_\rho^N(A_r) \geq \sigma_\rho^N(B_r)$$

for every $r \geq 0$.

We denote the standard Gaussian measure on $\mathbb{R}^n$ by γ, the one-dimensional standard Gaussian measure by γ^1 and the cummulative distribution function of γ^1 by

$$\Phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-t^2/2} \mathrm{d}t.$$

The Gaussian measure γ can be considered as the limit of $\sigma_{\sqrt{N}}^N$ for $N \to \infty$ in the following sense: If $\pi_{N+1,n} \colon \mathbb{R}^{N+1} \to \mathbb{R}^n$ denotes the projection onto the first n-components of a vector in $\mathbb{R}^{N+1}$ $(N \geq n)$ then we have

$$\lim_{N\to\infty} \sigma_{\sqrt{N}}^N \left(\pi_{N+1,n}^{-1}(A) \cap S_{\sqrt{N}}^N \right) = \gamma(A)$$

for all measurable $A \subset \mathbb{R}^n$. A proof of this fact can be found in [8].

As geodesic balls on S_ρ^N arise as the intersection of the sphere with half-spaces it is not difficult to believe that in the Gaussian isoperimetric inequality, half-spaces will fill the role of balls and geodesic balls in $\mathbb{R}^n$ and on S_ρ^N, respectively. A half-space H in $\mathbb{R}^n$ is a set of the form

$$H = \{x \in \mathbb{R}^n; \langle x, u \rangle \geq a\}$$

for some $a \in \mathbb{R}$ and a unit vector $u \in \mathbb{R}^n$.

Therefore, the following result can be seen as an infinite dimensional version of Theorem 1.1.

Theorem 1.2. *If $A \subset \mathbb{R}^n$ is measurable and H is a half-space with $\gamma(A) \geq \gamma(H)$ then*

$$\gamma(A_r) \geq \gamma(H_r)$$

for every $r \geq 0$.

Since $\gamma(\{x \in \mathbb{R}^n; \langle x, u \rangle \geq a\}) = \Phi(a)$ the theorem can be stated equivalently as

$$\gamma(A_r) \geq \Phi(\Phi^{-1}(\gamma(A)) + r) \tag{1.2}$$

for every $r \geq 0$. This formulation will immediately imply Theorem 2.2, a result about the concentration of the Gaussian measure.

Using Minkowski's formula (1.1) as a motivation to define the Gaussian surface area of a measurable set $A \subset \mathbb{R}^n$ via

$$\gamma^+(A) := \liminf_{r\downarrow 0} \frac{\gamma(A_r) - \gamma(A)}{r}$$

where $A_r := \{x \in \mathbb{R}^n; |x-y| \leq r \text{ for some } y \in A\}$ is the r-extension of A we can state the Gaussian isoperimetric inequality in the following form.

Theorem 1.3 (Gaussian isoperimetric inequality)**.** *If $A \subset \mathbb{R}^n$ is a measurable subset then*

$$\gamma^+(A) \geq \gamma^1(\Phi^{-1}(\gamma(A))) = I(\gamma(A)).$$

Here, we used the notation $I(x) := \gamma^1(\Phi^{-1}(x))$. In section 4 we will deduce this inequality from a generalization first proved by Borell in [2].

For n-dimensional standard Gaussian vectors X and Y with $\mathbb{E}X_iY_j = \rho\delta_{ij}$ and $0 < \rho < 1$ he introduced the Gaussian noise stability $\Pr_\rho(X \in A, Y \in A)$, i.e. the probability that X and Y lie both in a measurable set $A \subset \mathbb{R}^n$ and showed that it fulfills

$$\Pr_\rho(X \in A, Y \in A) \leq \Pr_\rho(X_1 \leq a, Y_1 \leq a) \tag{1.3}$$

for $0 < \rho < 1$ and $a := \Phi^{-1}(\gamma(A))$. Note that a is chosen such that $\gamma(A) = \gamma(\{x \in \mathbb{R}^n; x_1 \leq a\})$. This means that half-spaces maximize the Gaussian noise stability among all measurable sets with the same Gaussian measure. In section 4 we will see that Theorem 1.3 follows in the limit $\rho \to 1$ indeed from this inequality or more precisely from the Gaussian noise sensitivity, i.e. the inequality

$$\mathrm{Pr}_\rho(X \in A, Y \notin A) \geq \mathrm{Pr}_\rho(X_1 \leq a, Y_1 \geq a) \tag{1.4}$$

which is equivalent to (1.3). The proof follows a semigroup argument proposed by Ledoux in [7].

In section 4 we will verify that $\mathrm{Pr}_\rho(X \in A, Y \in A) = \mathcal{S}_{\rho^2}(A)$ for every measurable subset $A \subset \mathbb{R}^n$ where

$$\mathcal{S}_\rho(A) = \mathbb{P}\left(\sqrt{\rho}X + \sqrt{1-\rho}Y \in A, \sqrt{\rho}X + \sqrt{1-\rho}Y' \in A\right) \tag{1.5}$$

for $0 \leq \rho \leq 1$ where X, Y and Y' are independent standard Gaussian vectors in $\mathbb{R}^n$. In the following we will refer to this quantity as the *Gaussian noise stability.* With this definition, (1.3) can be formulated as in the following theorem.

Theorem 1.4 (Gaussian noise stability). *If $A, H \subset \mathbb{R}^n$ are measurable subsets such that H is a half-space with $\gamma(A) = \gamma(H)$ and $0 \leq \rho \leq 1$ then*

$$\mathcal{S}_\rho(A) \leq \mathcal{S}_\rho(H).$$

Definition (1.5) gives rise to the following generalization. For a measurable subset $A \subset \mathbb{R}^n$ we define the r-stability of A for $r > 1$ and $0 < \rho < 1$ through

$$\mathcal{S}_\rho^r(A) := \mathbb{E}\left[\mathbb{P}\left(\sqrt{\rho}X + \sqrt{1-\rho}Y \in A \middle| X\right)^r\right].$$

We show in section 4 that this quantitiy introduced by E. Mossel is a generalization of the Gaussian noise stability. Indeed, we will prove that $\mathcal{S}_\rho(A) = \mathcal{S}_\rho^2(A)$ for every measurable subset $A \subset \mathbb{R}^n$ and $0 < \rho < 1$. Therefore, Theorem 1.4 and thus (1.3) will follow from the next result which is the main result of these notes.

In order to describe the equality cases in the theorem properly, we need the following notation. For a measurable subset $B \subset \mathbb{R}^n$ we define its center of mass with respect to the Gaussian measure $v(B) = (v_i(B))_{i=1}^n$ with the components

$$v_i(B) = \int_B x_i \gamma(x) \mathrm{d}x.$$

If $v_i(B) = 0$ for $i = 1, \ldots, n$ then we set $v(B) = e_1$. Moreover, we set $q(B) := \|v(B)\|_2$, i.e. the distance of this center of mass from the origin. This enables us to associate the half-space

$$H(B) := \{x \in \mathbb{R}^n; \langle v(B), x\rangle \geq \alpha\} \tag{1.6}$$

to B where α is chosen such that $\gamma(B) = \gamma(H(B))$. The importance of $H(B)$ is caused by the fact that the symmetric difference of B and $H(B)$ is a null set if $\mathcal{S}_\rho^r(B)$ agrees with the r-stability of a half-space with Gaussian measure $\gamma(B)$. More precisely, the following generalization of Theorem 1.4 holds true.

Theorem 1.5. *For a measurable subset $A \subset \mathbb{R}^n$, $0 < \rho < 1$ and $r > 1$ we have*

$$\mathcal{S}_\rho^r(H(A)) \geq \mathcal{S}_\rho^r(A). \tag{1.7}$$

Equality holds if and only if the symmetric difference of A and $H(A)$ has measure zero.

Isaksson and Mossel proved (1.7) first in [6] and Neeman established the equality case in [10]. We will give a proof based on stochastic calculus due to Eldan [4].

We conclude this section with an outline of the remaining contents of this note. The next section contains some notations and remarks as well as an application of the Gaussian isoperimetric inequality to concentration of the Gaussian measure. In the third section, we give a proof of the main result of this note, Theorem 1.5, based on techniques from stochastic calculus. The following section consist of a proof of the Gaussian isoperimetric inequality which uses Theorem 1.5. In the end of this work, we collect some abstract auxiliary results employed in the previous sections.

2. Notations and Remarks

In this section, we fix some notation frequently used in the sequel and we make some remarks about the connection between a measurable set B and its half-space $H(B)$.

We denote the density of the Gaussian distribution with expectation value $v \in \mathbb{R}^n$ and covariance matrix $\sigma^2 \mathbb{1}$ for $\sigma > 0$ by

$$\gamma_{v,\sigma}(x) := \frac{1}{\sigma^n (2\pi)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}|x-v|^2\right).$$

We use the abbreviations $\gamma(x) := \gamma_{0,1}(x)$ for the density of the n-dimensional standard normal distribution and $\gamma^1(x) := \gamma_{0,1}(x)$ for the density of the one-dimensional standard normal distribution. By abuse of notation we write

$$\gamma(A) := \int_A \gamma(x)\mathrm{d}x$$

for a measurable subset $A \subset \mathbb{R}^n$ and $\mathrm{d}\gamma$ for the measure $\gamma_{0,1}(x)\mathrm{d}x$ on $\mathbb{R}^n$ and γ^1 for the one-dimensional version, respectively.

Moreover, we define for $s \in \mathbb{R}$ the auxiliary function $\tilde{q}$ through

$$\tilde{q}(s) := -\int_{-\infty}^{\Phi^{-1}(s)} t\mathrm{d}\gamma^1(t).$$

For the next remark, we remind the definition of $H(B)$ in (1.6).

Remark 2.1. *For a half-space $H = \{x; \langle v(B), x\rangle \geq \alpha\}$ we have*

$$q(H(B)) = \int_{\alpha/q(B)}^{\infty} t d\gamma^1(t) = \tilde{q}(\gamma(B)).$$

Proof. Take $S \in \mathrm{SO}(n)$ such that $Sv(B) = q(B)e_1$. Then

$$SH(B) = \{y; y_1 q(B) \geq \alpha\}.$$

Thus,

$$\int_{H(B)} x\gamma(x)\mathrm{d}x = \int_{SH(B)} S^{-1}x\gamma(x)\mathrm{d}x = S^{-1}\int_{SH(B)} x\gamma(x)\mathrm{d}x = S^{-1}e_1 \int_{\alpha/q(B)} t\gamma^1(t)\mathrm{d}t = \frac{v(B)}{q(B)} \int_{\alpha/q(B)} t\gamma^1(t)\mathrm{d}t.$$

For the second equality, we observe that similarly as before we get $\gamma(H(B)) = \Phi(-\alpha/q(B))$. Therefore, since $\gamma(B) = \gamma(H(B))$ by definition of $H(B)$ we have

$$\tilde{q}(\gamma(B)) = -\int_{-\infty}^{-\alpha/q(B)} t\gamma^1(t)\mathrm{d}t = \int_{\alpha/q(B)}^{\infty} t\gamma^1(t)\mathrm{d}t = \left|\int_{H(B)} x\gamma(x)\mathrm{d}x\right|.$$

□

Concentration of measure

Here, we give an application of the Gaussian isoperimetric inequality in the version (1.2). The following result states that a function is concentrated around its mean with respect to the standard Gaussian measure, i.e. the Gaussian measure of the set where the function is far away from its mean is small.

Theorem 2.2. *If $f: \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant L and M is its median with respect to γ then*

$$\gamma\left(\{x \in \mathbb{R}^n; f(x) \geq M + t\}\right) \leq \Phi\left(-\frac{t}{L}\right) \leq \exp\left(-\frac{t^2}{2L^2}\right)$$

for every $t \geq 0$.

Proof. Set $A := \{x \in \mathbb{R}^n; f(x) \leq M\}$. Thus, $\gamma(A) = 1/2$ since M is the median of f. Since f has Lipschitz constant L we have $|f(x) - M| \leq L\mathrm{dist}(x, A)$ where $\mathrm{dist}(x, A) = \inf\{|x-y|; y \in A\}$. Thus, $A_{t/L} \subset \{x \in \mathbb{R}^n; f(x) \leq M + t\}$. Using $\Phi^{-1}(\gamma(A)) = \Phi^{-1}(1/2) = 0$ and (1.2) we get

$$\gamma(\{x \in \mathbb{R}^n; f(x) \leq M + t\}) \geq \gamma(A_{t/L}) \geq \Phi(t/L).$$

The first inequality follows from this estimate and the second inequality is a standard estimate of Φ. □

3. Proof of the r-stability inequality

In this section we give a proof of the inequality in Theorem 1.5 which is based on [4]. For a proof of the equality case, we refer to [4].

Let W_t be a standard Brownian motion in $\mathbb{R}^n$ adapated to the filtration $(\mathcal{F}_t)_{t\in[0,1]}$ on the probability space $(\Omega_1, \Sigma_1, P_1)$. We define the stochastic process

$$S_t := \mathbb{P}(W_1 \in A|\mathcal{F}_t) \tag{3.1}$$

for $t \in [0,1]$. For $t \in [0,1)$ we have the relation

$$S_t = \mathbb{P}(W_1 \in A|W_t) = \int_A \gamma_{W_t,\sqrt{1-t}}(x)\mathrm{d}x. \tag{3.2}$$

This shows that the process S_t interpolates between $\gamma(A)$ and $\mathbb{1}_A(W_1)$ (compare (3.10)).

Using the substitution $y = (1-t)^{-1/2}(x - W_t)$ and the notation

$$A_t := \frac{A - W_t}{\sqrt{1-t}}$$

we can rewrite (3.2) as

$$S_t = \int_{A_t} \gamma(x)\mathrm{d}x = \gamma(A_t). \tag{3.3}$$

Corresponding to the definition (3.1) we introduce as analogue for the half-space $H(A)$ the stochastic process

$$Q_t := \mathbb{P}(\tilde{W}_1 \in H(A)|\tilde{\mathcal{F}}_t) \tag{3.4}$$

for a standard Brownian motion $\tilde{W}_t$ in $\mathbb{R}^n$ adapted to the filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,1]}$ on the probability space $(\Omega_2, \Sigma_2, P_2)$. Similarly as for S_t, the process Q_t is an interpolation between $\gamma(H(A))$ and $\mathbb{1}_{H(A)}(\tilde{W}_1)$.

Our first goal is an equivalent formulation of Theorem 1.5 given in Proposition 3.2. Therefore, we need the following technical Lemma which uses the notion of the quadratic variation.

For an Itô process X_t with $\mathrm{d}X_t = v(t,\omega)\mathrm{d}B_t(\omega)$ where B_t is a Brownian motion on $\mathbb{R}^n$ the *quadratic variation* $[X]_t$ of X_t is the stochastic process

$$[X]_t := \int_0^t |v_s|^2 \mathrm{d}s$$

or in differential notation

$$\mathrm{d}[X]_t = |v_t|^2 \mathrm{d}t. \tag{3.5}$$

Lemma 3.1. *We have*

$$d[S]_t = (1-t)^{-1} q^2(A_t)\,dt, \tag{3.6}$$

$$d[Q]_t = (1-t)^{-1} \tilde{q}^2(Q_t)\,dt. \tag{3.7}$$

Proof. Lemma A.2 with $\phi = \mathbb{1}_A$ gives

$$\begin{aligned}
\mathrm{d}S_t &= \mathrm{d}\int_A F_t(x)\mathrm{d}x \\
&= (1-t)^{-1}\left\langle \int_A (x - W_t)F_t(x)\mathrm{d}x, \mathrm{d}W_t \right\rangle \\
&= \frac{(1-t)^{-1}}{(2\pi(1-t))^{n/2}}\left\langle \int_A (x - W_t)\exp\left(-\frac{1}{2(1-t)}|x - W_t|^2\right)\mathrm{d}x, \mathrm{d}W_t \right\rangle \\
&= (1-t)^{-1/2}\left\langle \int_{A_t} y\gamma(y)\mathrm{d}y, \mathrm{d}W_t \right\rangle \\
&= (1-t)^{-1/2}\sum_{i=1}^n \left(\int_{A_t} x_i \mathrm{d}\gamma(x)\right)\mathrm{d}W_t^i
\end{aligned} \tag{3.8}$$

where we used the notation of Lemma A.2 and the substitution $y = (1-t)^{-1/2}(x - W_t)$ in the last step.

Therefore, since S_t is an Itô process by (3.8) the relation in (3.6) follows immediately from (3.5).

Moreover, as in (3.3) we get

$$Q_t = \gamma(H_t)$$

where

$$H_t := \frac{H(A) - \tilde{W}_t}{\sqrt{1-t}}.$$

Applying Lemma A.2 with $\phi = \mathbb{1}_{H(A)}$ and proceeding as in (3.8) yields

$$\mathrm{d}Q_t = (1-t)^{-1/2}\left\langle \int_{H_t} x\mathrm{d}\gamma(x), \mathrm{d}\tilde{W}_t \right\rangle$$

which implies in the same ways as in (3.6) that

$$\mathrm{d}[Q]_t = (1-t)^{-1} q(H_t)^2\,\mathrm{d}t = (1-t)^{-1}\tilde{q}^2(Q_t)\mathrm{d}t$$

where we used $q(H_t) = \tilde{q}(\gamma(H_t)) = \tilde{q}(Q_t)$ by Remark 2.1. ☐

Proposition 3.2. *The assertion of Theorem 1.5 is equivalent to*

$$\mathbb{E}\left[\int_0^\rho Q_t^{r-2} d[Q]_t\right] \geq \mathbb{E}\left[\int_0^\rho S_t^{r-2} d[S]_t\right] \tag{3.9}$$

with equality if and only if $\gamma(A\Delta H(A)) = 0$.

Proof. Applying Itô's formula to (3.8) yields

$$\begin{aligned}\mathrm{d}S_t^r &= rS_t^{r-1}\mathrm{d}S_t + \frac{1}{2}r(r-1)S_t^{r-2}(\mathrm{d}S_t)^2 \\ &= rS_t^{r-1}\mathrm{d}S_t + \frac{1}{2}r(r-1)S_t^{r-2}(1-t)^{-1}\sum_{i=1}^n\left(\int_{A_t} x_i\mathrm{d}\gamma(x)\right)^2(\mathrm{d}W_t^i)^2 = rS_t^{r-1}\mathrm{d}S_t + \frac{1}{2}r(r-1)S_t^{r-2}\mathrm{d}[S]_t\end{aligned}$$

where we used $\mathrm{d}W_t^i\mathrm{d}W_t^j = \delta_{ij}\mathrm{d}t$ in the last two steps.

Integrating and taking the expectation of the last equation implies

$$\mathcal{S}_\rho^r(A) = \mathbb{E}S_\rho^r = S_0^r + \mathbb{E}\left[\int_0^\rho \mathrm{d}S_t^r\right] = S_0^r + \frac{1}{2}r(r-1)\mathbb{E}\left[\int_0^\rho S_t^{r-2}\mathrm{d}[S]_t\right]$$

where we used in the second step

$$\mathbb{E}\left[\int_0^\rho rS_t^{r-1}\mathrm{d}S_t\right] = \mathbb{E}\left[\sum_{i=1}^n\int_0^\rho rS_t^{r-1}(1-t)^{-1/2}\left(\int_{A_t} x_i\mathrm{d}\gamma(x)\right)\mathrm{d}W_t^i\right] = 0$$

because of Theorem 3.2.1 (iii) in [11]. In particular, for $r=2$ we get $S_\rho(A) = \gamma(A)^2 + \mathbb{E}[S]_\rho$.

As before, we get

$$\mathcal{S}_\rho^r(H(A)) = \mathbb{E}[Q_\rho^r] = Q_0^r + \frac{1}{2}r(r-1)\mathbb{E}\left[\int_0^\rho Q_t^{r-2}\mathrm{d}[Q]_t\right].$$

Since $Q_0 = \gamma(H(A)) = \gamma(A) = S_0$ the equivalence of Theorem 1.5 and (3.9) follows. ☐

Our next goal is to couple the processes S_t and Q_t such that they are both time changed versions of the same Brownian motion. The first step is to show that by enlarging the probability spaces we can assume that S_t and Q_t are time changed versions of two Brownian motions. By the definitions (3.1) and (3.4) the stochastic processes S_t and Q_t are martingales with respect to the filtrations $(\mathcal{F}_t)_t$, $(\tilde{\mathcal{F}}_t)_t$, respectively. Therefore, we can apply Theorem A.3 to $S_t - S_0$ and $Q_t - Q_0$.

Applying Theorem A.3 to $(\Omega_1, \Sigma_1, P_1)$ and the process $S_t - S_0$ yields a probability space $(\Omega_1', \Sigma_1', P_1')$ which is an enlargement of $(\Omega_1, \Sigma_1, P_1)$ and a process $B(t)$ such that $B(t) - S_0$ is a standard Brownian motion in $\mathbb{R}$ and

$$S_t = B([S]_t)$$

for $t \in [0,1]$.

Moreover, applying Theorem A.3 to $(\Omega_2, \Sigma_2, P_2)$ and the process $Q_t - Q_0$ yields a probability space $(\Omega_2', \Sigma_2', P_2')$ which is an enlargement of $(\Omega_2, \Sigma_2, P_2)$ and a process $\tilde{B}(t)$ such that $\tilde{B}(t) - Q_0$ is a standard Brownian motion in $\mathbb{R}$ and

$$Q_t = \tilde{B}([Q]_t)$$

for $t \in [0,1]$.

The next Lemma is used in the proof of the validity of (3.9).

Lemma 3.3. *We have almost surely*

$$T_f := [S]_1 = \min\{t > 0; B(t) \in \{0,1\}\}, \qquad \tilde{T}_f := [Q]_1 = \min\{t > 0; \tilde{B}(t) \in \{0,1\}\}.$$

Proof. Due to $0 < \gamma(A) < 1$ and the strict positivity of the density of the Gaussian measure we have $0 < S_t < 1$ and $0 < Q_t < 1$ for $0 \le t < 1$. Since

$$S_1 = \mathbb{P}(W_1 \in A|\mathcal{F}_1) = \mathbb{E}[\mathbb{1}_A(W_1)|W_1] = \mathbb{1}_A(W_1) \tag{3.10}$$

we get $S_1 \in \{0,1\}$ and similarly $Q_1 \in \{0,1\}$ and therefore $B([S]_1) = S_1 \in \{0,1\}$, $\tilde{B}([Q]_1) = Q_1 \in \{0,1\}$. As $[S]_t < [S]_s$ and $[Q]_t < [Q]_s$ for $t < s$ the assertion follows. □

In the second step, we can now introduce a probability space such that the two Brownian motions $B(t)$ and $\tilde{B}(t)$ agree.

Applying Theorem A.8 to the probability spaces $(\Omega_1', \Sigma_1', P_1')$ and $(\Omega_2', \Sigma_2', P_2')$ and the processes $B(t) - S_0$ and $\tilde{B}(t) - Q_0 = \tilde{B}(t) - S_0$ we can assume that the processes $W_t, \tilde{W}_t, S_t, Q_t, B(t)$ and $\tilde{B}(t)$ are defined on a common probability space (Ω, Σ, P) such that $B(t) = \tilde{B}(t)$ for all $t \ge 0$. In the following we will therefore write $B(t)$ instead of $\tilde{B}(t)$.

The functions τ_1 and τ_2 in the next Lemma will provide appropriate changes of variables for the integrals in (3.9) in order to prove this inequality.

Lemma 3.4. *The functions $T_1(t) := [S]_t$ and $T_2(t) := [Q]_t$ are invertible and their inverse functions $\tau_1(T)$ and $\tau_2(T)$ are differentiable for $0 < T < T_f$ with*

$$\tau_1'(T) = (1 - \tau_1(T))q^{-2}(A_{\tau_1(T)}), \qquad \tau_2'(T) = (1 - \tau_2(T))\tilde{q}^{-2}(B(T)).$$

Proof. By (3.6) and (3.7) the processes $[S]_t$ and $[Q]_t$ are almost surely continuous and strictly increasing for $t \in (0,1)$. Therefore, the inverse functions exist on $[0, T_f)$. Moreover, (3.6) implies

$$T_1'(\tau_1(T)) = (1 - \tau_1(T))^{-1} q^2(A_{\tau_1(T)})$$

for $T \in (0, T_f)$. As $\tau_1(T) \in (0,1)$ and $q(A_{\tau_1(T)}) > 0$ almost surely for $T \in (0, T_f)$ we have $T_1'(\tau_1(T)) \neq 0$ and thus $\tau_1(T)$ differentiable for $T \in (0, T_f)$. Similarly, we get the differentiability of $\tau_2(T)$ for $T \in (0, T_f)$.

Applying the formula for the derivative of the inverse function yields

$$\tau_1'(T) = (1 - \tau_1(T))q^{-2}(A_{\tau_1(T)}), \qquad \tau_2'(T) = (1 - \tau_2(T))\tilde{q}^{-2}(B(T))$$

for $T \in (0, T_f)$ where we used for the conclusion of the second relation that

$$Q_{\tau_2(T)} = B([Q]_{\tau_2(T)}) = B(T) \tag{3.11}$$

as $T_2(t) = [Q]_t$ and $\tau_2(T)$ are inverse functions. □

The following Lemma combined with changes of variables via τ_1 and τ_2 will allow us to prove (3.9) which will conclude the proof of Theorem 1.5 by Proposition 3.2.

Lemma 3.5. *We have*

$$[S]_t \le [Q]_t$$

for all $t \in [0,1]$.

The proof of this Lemma is based on Lemma 3.6 which we show at the end of this section.

Proof. We define the functions

$$\omega_1(T) := -\log(1 - \tau_1(T)), \qquad \omega_2(T) := -\log(1 - \tau_2(T)).$$

Lemma 3.4 implies

$$\omega_1'(T) = q^{-2}(A_{\tau_1(T)}), \qquad \omega_1'(T) = \tilde{q}^{-2}(B(T)).$$

Applying Lemma 3.6 yields

$$q(A_{\tau_1(T)}) \leq \tilde{q}(\gamma(A_{\tau_1(T)})) = \tilde{q}(B(T))$$

where we used $\gamma(A_t) = S_t = B([S]_t)$, i.e. $\gamma(A_{\tau_1(T)}) = B(T)$, in the last step. Therefore, we get

$$\omega_1'(T) - \omega_2'(T) = q^{-2}(A_{\tau_1(T)}) - \tilde{q}^{-2}(B(T)) \geq 0$$

which implies

$$\omega_1(T) \geq \omega_2(T) \tag{3.12}$$

for $T \in [0, T_f)$ since $\omega_1(0) = \omega_2(0)$ by $[S]_0 = 0 = [Q]_0$. Using (3.12) we get $\tau_1(T) \geq \tau_2(T)$ for $T \in [0, T_f)$. Thus,

$$[S]_t \leq [Q]_t$$

for $t \in [0, 1]$. □

Now, we will prove the inequality Theorem 1.5 by reducing (3.9) to another inequality which immediately follows from Lemma 3.5. Note that for a proof of the equality case we refer to [4].

Proof of Theorem 1.5. By Proposition 3.2 it suffices to show (3.9). Using the substitution $t = \tau_2(T)$, Lemma 3.4 and (3.11) we get

$$\int_0^\rho Q_t^{r-2} \mathrm{d}[Q]_t = \int_0^\rho Q_t^{r-2} (1-t)^{-1} \tilde{q}^2(Q_t) \mathrm{d}t = \int_0^{[Q]_\rho} B(T)^{r-2} \mathrm{d}T.$$

Similarly, the substitution $t = \tau_1(T)$ and Lemma 3.4 yields

$$\int_0^\rho S_t^{r-2} \mathrm{d}[S]_t = \int_0^{[S]_\rho} B(T)^{r-2} \mathrm{d}T.$$

Therefore, the inequality (3.9) is equivalent to

$$\mathbb{E}\left[\int_0^{[Q]_\rho} B(T)^{r-2} \mathrm{d}T\right] \geq \mathbb{E}\left[\int_0^{[S]_\rho} B(T)^{r-2} \mathrm{d}T\right]. \tag{3.13}$$

As $B(T) > 0$ for $0 < T < T_f$ by Lemma 3.3 (3.13) holds true because of Lemma 3.5. This concludes the proof of Theorem 1.5. □

The following lemma is a crucial ingredient of the proof of the Lemma 3.5. It establishes the intuitive statement that half-spaces maximize the distance between the origin and the center of mass with respect to the Gaussian measure.

Lemma 3.6. *If $B \subset \mathbb{R}^n$ is a measurable set then*

$$q(B) \leq \tilde{q}(\gamma(B)) = q(H(B)). \tag{3.14}$$

Equality holds if and only if $\gamma(B \Delta H(B)) = 0$.

The following proof is based on the next Lemma.

Lemma 3.7. *For a measurable function $m\colon \mathbb{R} \to [0, 1]$ we have*

$$\left|\int_{\mathbb{R}} x m(x) d\gamma^1(x)\right| \leq \tilde{q}\left(\int_{\mathbb{R}} m(x) d\gamma^1(x)\right)$$

with equality if and only if there is $\alpha \in \mathbb{R}$ such that $m(x) = \mathbb{1}_{\{y \geq \alpha\}}(x)$ for almost every $x \in \mathbb{R}$ with

$$\Phi(-\alpha) = \int_{\mathbb{R}} m(x) d\gamma^1(x).$$

Proof. First, by possibly replacing $m(x)$ by $m(-x)$ we can assume that

$$\int_{\mathbb{R}} xm(x)\mathrm{d}\gamma^1(x) \geq 0.$$

We set $h(x) \coloneqq \mathbb{1}_{\{y\geq\alpha\}}(x)$ where α is chosen such that

$$\int_{\mathbb{R}} m(x)\mathrm{d}\gamma^1(x) = \int_{\mathbb{R}} h(x)\mathrm{d}\gamma^1(x). \tag{3.15}$$

Note that the last integral equals $\Phi(-\alpha)$. In particular,

$$\tilde{q}\left(\int_{\mathbb{R}} m(x)\mathrm{d}\gamma^1(x)\right) = -\int_{-\infty}^{-\alpha} t\gamma^1(t)\mathrm{d}t = \int_{\alpha}^{\infty} t\gamma^1(t)\mathrm{d}t = \int_{\mathbb{R}} th(t)\mathrm{d}\gamma^1(t).$$

Therefore, the statement of the Lemma is equivalent to show that

$$0 \leq \int_{\mathbb{R}} t(h(t)-m(t))\mathrm{d}\gamma^1(t) = \int_{\mathbb{R}} (t-\alpha)(h(t)-m(t))\mathrm{d}\gamma^1(t)$$

where in the last step we used (3.15). But the integrand of the last integral is non-negative since $(t-\alpha)$ has the same sign as $(h(t)-m(t))$ as $0 \leq m(t) \leq 1$ for all $t \in \mathbb{R}$. Moreover, the integral vanishes if and only if $m(t) = h(t)$ for almost all $t \in \mathbb{R}$. ☐

Proof of Lemma 3.6. If $q(B) = 0$ then the inequality is trivially fulfilled. Assume now $q(H(B)) = q(B) = 0$ then $H(B) = \mathbb{R}^n$ and therefore $\gamma(B) = \gamma(H(B)) = 1$ which implies $\gamma(B\Delta H(B)) = 0$. For $q(B) > 0$ we define

$$\theta \coloneqq q(B)^{-1}\int_B x\gamma(x)\mathrm{d}x.$$

Thus, we have

$$\left|\int_B \langle\theta, x\rangle\gamma(x)\mathrm{d}x\right| = q(B)^{-1}\sum_{i=1}^{n} v_i(B)\int_B x_i\gamma(x)\mathrm{d}x = q(B)^{-1}\|v(B)\|_2^2 = q(B).$$

For $p\colon \mathbb{R}^n \to \mathbb{R}, x \mapsto \langle\theta, x\rangle$ we define the pushforward measure of $\gamma|_B$ as $\mu \coloneqq p_{\#}\gamma(\cdot \cap B)$. Then the formula for the change of variables implies

$$\int_B \langle\theta, x\rangle\gamma(x)\mathrm{d}x = \int_B p(x)\mathrm{d}x = \int_{\mathbb{R}} t\mathrm{d}\mu(t) = \int_{\mathbb{R}} tm(t)\mathrm{d}\gamma^1(t)$$

where we introduced the function $m(t) = \frac{\mathrm{d}\mu}{\mathrm{d}\gamma^1}(t)$. Therefore, Lemma 3.7 yields

$$q(B) = \left|\int_{\mathbb{R}} tm(t)\mathrm{d}\gamma^1(t)\right| \leq \tilde{q}\left(\int_{\mathbb{R}} m(t)\mathrm{d}\gamma^1(t)\right) = \tilde{q}(\mu(\mathbb{R})) = \tilde{q}(\gamma(B)).$$

Thus, the inequality follows from Remark 2.1.

If $\gamma(B\Delta H(B)) = 0$ we clearly have $q(B) = q(H(B))$. Assume that we have equality in (3.14). Then by Lemma 3.7 we have

$$\gamma(B) = \gamma(p^{-1}([\alpha,\infty) \cap B))$$

with $\Phi(-\alpha) = \gamma(p^{-1}([\alpha,\infty)))$.

We assume equality in (3.14) and set $H_{\theta,\alpha} \coloneqq \{x; \langle x,\theta\rangle \geq \alpha\} = p^{-1}([\alpha,\infty))$. First, we prove $\gamma(H_{\theta,\alpha}\Delta B) = 0$ and afterwards $H_{\theta,\alpha} = H(B)$. As in the proof of Remark 2.1 one can show $\gamma(H_{\theta,\alpha}) = \Phi(-\alpha)$. Since $\mathbb{1}_{s\geq\alpha}(t) = m(t) = \frac{\mathrm{d}\mu}{\mathrm{d}\gamma^1}(t)$ we have

$$\gamma(H_{\theta,\alpha} \cap B) = \gamma(p^{-1}([\alpha,\infty))) = \int_{\alpha}^{\infty} \mathrm{d}\gamma^1 = \Phi(-\alpha) = \gamma(H_{\theta,\alpha})$$

and

$$\gamma(B) = \gamma(\mathbb{R}^n \cap B) = \mu(\mathbb{R}) = \mu([\alpha,\infty)) = \gamma(H_{\theta,\alpha} \cap B).$$

Thus, $\gamma(H_{\theta,\alpha}\Delta B)=0$. Moreover, writing $H(B)=\{x;\langle x,v(B)\rangle\geq\beta\}$ we get by Remark 2.1

$$\Phi\left(-\frac{\beta}{q(B)}\right)=\gamma(H(B))=\gamma(B)=\gamma(H_{\theta,\alpha})=\Phi(-\alpha)$$

i.e. $\alpha=\beta/q(B)$ and therefore $H(B)=H_{\theta,\alpha}$.

□

4. Proof of the Gaussian Isoperimetric Inequality

In this section, we derive the Gaussian Isoperimetric Inequality, Theorem 1.3, from the Gaussian noise stability, Theorem 1.4. First, we will prove that the latter result is a consequence of Theorem 1.5.

Proof of Theorem 1.4. In order to show $\mathcal{S}_\rho(A)=\mathcal{S}_\rho^2(A)$ for all measurable sets $A\subset\mathbb{R}^n$ we first prove that

$$\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y\in A,\sqrt{\rho}X+\sqrt{1-\rho}Y'\in A\middle|X\right)=\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y\in A\middle|X\right)\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y'\in A\middle|X\right) \tag{4.1}$$

for independent standard Gaussian vectors X, Y and Y'. This means that $\sqrt{\rho}X+\sqrt{1-\rho}Y\in A$ and $\sqrt{\rho}X+\sqrt{1-\rho}Y'\in A$ are independent with respect to $\mathbb{P}(\cdot|X)$. Applying Lemma A.1 we get

$$\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y\in A,\sqrt{\rho}X+\sqrt{1-\rho}Y'\in A\middle|X\right)=g(X)$$

for $g(x):=\mathbb{E}\left[\mathbb{1}_A(x\sqrt{\rho}+Y\sqrt{1-\rho})\mathbb{1}_A(x\sqrt{\rho}+Y'\sqrt{1-\rho})\right]$. Since Y and Y' are independent we have $g(x)=\mathbb{E}\left[\mathbb{1}_A(x\sqrt{\rho}+Y\sqrt{1-\rho})\right]\mathbb{E}\left[\mathbb{1}_A(x\sqrt{\rho}+Y'\sqrt{1-\rho})\right]$. Hence, (4.1) follows from Lemma A.1.

Now, we compute

$$\begin{aligned}\mathcal{S}_\rho^q(A)&=\mathbb{E}\left[\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y\in A\middle|X\right)^2\right]\\&=\mathbb{E}\left[\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y\in A\middle|X\right)\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y'\in A\middle|X\right)\right]\\&=\mathbb{E}\left[\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y\in A\text{ and }\sqrt{\rho}X+\sqrt{1-\rho}Y'\in A\middle|X\right)\right]\\&=\mathbb{P}\left(\sqrt{\rho}X+\sqrt{1-\rho}Y\in A\text{ and }\sqrt{\rho}X+\sqrt{1-\rho}Y'\in A\right)\\&=\mathcal{S}_\rho(A)\end{aligned}$$

where we introduced a standard Gaussian vector Y' such that X,Y and Y' are independent in the second step and used (4.1) in the third step.

If X,Y are standard n-dimensional Gaussian vectors such that $\mathbb{E}X_iY_j=\rho\delta_{ij}$ then their joint density is given by

$$f(x,y):=\frac{1}{\sqrt{(2\pi)^{2n}(1-\rho^2)^n}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\langle x,x\rangle+\langle y,y\rangle-2\rho\langle x,y\rangle\right)\right).$$

Therefore, for a measurable set $A\subset\mathbb{R}^n$, using the substitution $z=(y-\rho x)/\sqrt{1-\rho^2}$ we get

$$\begin{aligned}\mathbb{P}\left(X\in A,Y\in A\right)&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathbb{1}_A(x)\mathbb{1}_A(y)f(x,y)\mathrm{d}x\mathrm{d}y\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathbb{1}_A(x)\mathbb{1}_A(\rho x+z\sqrt{1-\rho^2})\gamma(x)\gamma(z)\mathrm{d}x\mathrm{d}z\\&=\mathbb{P}\left(Z_1\in A,\rho Z_1+\sqrt{1-\rho^2}Z_2\in A\right)\end{aligned}\tag{4.2}$$

for some independent standard n-dimensional Gaussian vectors Z_1 and Z_2.

Replacing ρ by $\sqrt{\rho}$ we see that $\mathcal{S}_{\rho^2}(A)=\mathbb{P}(X\in A,Y\in A)=\Pr_\rho(X\in A,Y\in A)$. By the rotational invariance of the standard Gaussian measure we conclude that Theorem 1.4 follows from Theorem 1.5. □

Now, we turn to the proof of the Gaussian isoperimetric inequality. We show this result by following the reasoning in section 2.2 of [10] which is based on [7].

As in the proof of the Gaussian isoperimetric inequality by Bakry and Ledoux [1] we will use the Ornstein-Uhlenbeck operator semigroup to deduce the Gaussian isoperimetric inequality from the Gaussian noise stability. Therefore, we define for any $t \geq 0$ the operator $P_t \colon L^2(\mathbb{R}^n, \gamma) \to L^2(\mathbb{R}^n, \gamma)$ through

$$(P_t f)(x) = \int_{\mathbb{R}^n} f\left(xe^{-t} + y\sqrt{1 - e^{-2t}}\right) \mathrm{d}\gamma(y)$$

for almost every $x \in \mathbb{R}^n$. Futhermore, we introduce the operator $L \colon C_0^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \gamma)$ through

$$(Lf)(x) = \sum_{i=1}^n \left(\frac{\partial^2 f}{\partial x_i^2}(x) - x_i \frac{\partial f}{\partial x_i}(x) \right) = \Delta f(x) - \langle x, \nabla f(x) \rangle$$

for almost every $x \in \mathbb{R}^n$. Note that $(P_t)_{t \geq 0}$ is a semigroup, i.e. $P_0 = id$ and $P_t P_s = P_{t+s}$ for all $t, s \geq 0$, and the operator L is the generator of this semigroup, i.e. $P_t = e^{tL}$.

We collect some simple properties of the operators P_t and L in the following lemma.

Lemma 4.1. *It holds*

(i) For a smooth function f and $t \geq 0$ we have

$$\frac{d}{dt} P_t f = L P_t f.$$

(ii) The operator $P_t \colon L^2(\gamma) \to L^2(\gamma)$ is self-adjoint for every $t \geq 0$.

(iii) For a smooth function f we have

$$\nabla P_t f = e^{-t} P_t \nabla f.$$

Note that we write $\mathbb{E} f := \mathbb{E} f(X)$ for $f \in L^1(\gamma)$ and a standard n-dimesional Gaussian vector X, i.e.

$$\mathbb{E} f = \int_{\mathbb{R}^n} f(x) \mathrm{d}\gamma(x).$$

Using this notation and $\nabla \gamma(x) = -x\gamma(x)$, it is straight forward to check the following version of an integration by parts formula

$$\mathbb{E} g L f = \int_{\mathbb{R}^n} g L f \; \mathrm{d}\gamma = - \int_{\mathbb{R}^n} \langle \nabla g, \nabla f \rangle \; \mathrm{d}\gamma = -\mathbb{E} \langle \nabla g, \nabla f \rangle. \tag{4.3}$$

Proposition 4.2. *For smooth bounded functions $f, g \colon \mathbb{R}^n \to \mathbb{R}$ with $g \geq 0$ we have*

$$\mathbb{E} g P_t f - \mathbb{E} g f \leq \frac{\|g\|_\infty}{\sqrt{2\pi}} \arccos(e^{-t}) \mathbb{E} |\nabla f|. \tag{4.4}$$

Proof. We start the estimate with rewritting the right-hand side. Since

$$P_t f - f = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} P_s f \mathrm{d}s = \int_0^t L P_s f \mathrm{d}s$$

by Lemma 4.1 (i) we have the following relations for the right-hand side of the assertion

$$\mathbb{E} g P_t f - \mathbb{E} g f = \int_0^t \mathbb{E} g L P_s f \mathrm{d}s = - \int_0^t \mathbb{E} \langle \nabla g, \nabla P_s f \rangle \mathrm{d}s = - \int_0^t \mathbb{E} \langle \nabla P_s g, \nabla f \rangle \mathrm{d}s, \tag{4.5}$$

where we used (4.3) in the second step and Lemma 4.1 (ii) and (iii) in the third step. The definition of P_s, Lemma 4.1 (iii) and integration by parts, observing $\nabla \gamma(y) = -y\gamma(y)$, yield

$$\begin{aligned}
(\nabla P_s g)(x) =& e^{-s} \int_{\mathbb{R}^n} (\nabla g) \left(xe^{-s} + y\sqrt{1 - e^{-2s}}\right) \mathrm{d}\gamma(y) \\
=& \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \int_{\mathbb{R}^n} y g \left(xe^{-s} + y\sqrt{1 - e^{-2s}}\right) \mathrm{d}\gamma(y) \\
=& \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \mathbb{E} \left[Z g \left(xe^{-s} + Z\sqrt{1 - e^{-2s}}\right) \right]
\end{aligned}$$

for a standard n-dimensional Gaussian vector Z. Combining this formula with (4.5), we get

$$\mathbb{E}gP_tf - \mathbb{E}gf = -\int_0^t \frac{e^{-s}}{\sqrt{1-e^{-2s}}}\mathbb{E}\left[\langle Z_2, \nabla f(Z_1)\rangle g\left(Z_1e^{-s} + Z_2\sqrt{1-e^{-2s}}\right)\right] \mathrm{d}s \tag{4.6}$$

with two independent standard n-dimensional Gaussian vectors Z_1 and Z_2. The estimates $0 \le g \le \|g\|_\infty$ imply

$$\begin{aligned}\mathbb{E}\left[\langle Z_2, \nabla f(Z_1)\rangle g\left(Z_1e^{-s} + Z_2\sqrt{1-e^{-2s}}\right)\right] &\ge \|g\|_\infty \mathbb{E}\min\{0, \langle Z_2, \nabla f(Z_1)\rangle\} \\ &= \frac{1}{2}\mathbb{E}\left[\langle Z_2, \nabla f(Z_1)\rangle - |\langle Z_2, \nabla f(Z_1)\rangle|\right].\end{aligned}$$

As $\mathbb{E}\langle Z_2, a\rangle = 0$ for $a \in \mathbb{R}^n$ we have

$$\frac{1}{2}\mathbb{E}\left[\langle Z_2, a\rangle - |\langle Z_2, a\rangle|\right] = -\frac{1}{2}|a|\int_{\mathbb{R}^n} |x_1|\mathrm{d}\gamma(x) = -\frac{|a|}{\sqrt{2\pi}}.$$

Therefore, conditioning on Z_1 and applying Lemma A.1 we get

$$\begin{aligned}\mathbb{E}\left[\langle Z_2, \nabla f(Z_1)\rangle g\left(Z_1e^{-s} + Z_2\sqrt{1-e^{-2s}}\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\langle Z_2, \nabla f(Z_1)\rangle g\left(Z_1e^{-s} + Z_2\sqrt{1-e^{-2s}}\right)\Big| Z_1\right]\right] \\ &\ge -\frac{1}{\sqrt{2\pi}}\mathbb{E}|\nabla f(Z_1)|.\end{aligned}$$

Thus, (4.6) yields

$$\mathbb{E}gP_tf - \mathbb{E}gf \le \frac{\|g\|_\infty}{\sqrt{2\pi}}\mathbb{E}|\nabla f|\int_0^t \frac{e^{-s}}{\sqrt{1-e^{-2s}}}\mathrm{d}s = \frac{\|g\|_\infty}{\sqrt{2\pi}}\mathbb{E}|\nabla f|\int_1^{e^{-t}} \frac{1}{\sqrt{1-s^2}}\mathrm{d}s = \frac{\|g\|_\infty}{\sqrt{2\pi}}\mathbb{E}|\nabla f|\arccos(e^{-t}).$$

This concludes the proof. □

By approximating the characteristic function of a measurable set appropriately we get the following corollary of the previous result:

Corollary 4.3. *If $A \subset \mathbb{R}^n$ is a measurable subset and $0 < \rho < 1$ then*

$$\Pr_\rho(X \in A, Y \notin A) \le \frac{1}{\sqrt{2\pi}}\arccos(\rho)\gamma^+(A).$$

Proof. Taking $g = \mathbb{1}_A$, $f = -\mathbb{1}_A$ and $\rho = e^{-t}$ on the left-hand side of (4.4) we get

$$\mathbb{E}gP_tf - \mathbb{E}gf = -\mathbb{P}\left(X \in A, Z_1\rho + Z_2\sqrt{1-\rho^2} \in A\right) + \gamma(A) = \Pr_\rho(X \in A, Y \notin A) \tag{4.7}$$

for some independent standard Gaussian vectors Z_1 and Z_2. Here, in the last step we used (4.2).

For $\varepsilon > 0$ we take $f_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}$ a smooth function such that $f|_A = 1$, $f|_{A_\varepsilon^c} = 0$ and $|\nabla f_\varepsilon| \le 1 + \varepsilon^{-1}$. Clearly,

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}f_\varepsilon P_t f_\varepsilon - \mathbb{E}f_\varepsilon^2 = \mathbb{E}\mathbb{1}_A P_t \mathbb{1}_A - \mathbb{E}\mathbb{1}_A = \Pr_\rho(X \in A, Y \notin A)$$

where we used $\gamma(A_\varepsilon \backslash A) \to 0$ for $\varepsilon \downarrow 0$ in the first step and (4.7) in the second step. Moreover,

$$\liminf_{\varepsilon\downarrow 0} \mathbb{E}|\nabla f_\varepsilon| \le \liminf_{\varepsilon\downarrow 0}\left(1+\varepsilon^{-1}\right)\gamma(A_\varepsilon\backslash A) = \gamma^+(A).$$

Thus, the assertion follows from (4.4). □

Lemma 4.4. *If $a \in \mathbb{R}$ then*

$$\lim_{\rho\uparrow 1}\frac{\Pr_\rho(X_1 \le a, Y_1 \ge a)}{\arccos(\rho)} = \frac{\gamma^1(a)}{\sqrt{2\pi}}.$$

Proof. Note that

$$\Pr_\rho(X_1 \le a, Y_1 \ge a) = \mathbb{P}(Z_1 \le a, \rho Z_1 + \sqrt{1-\rho^2}Z_2 \ge a) \tag{4.8}$$

for independent standard Gaussian variables Z_1 and Z_2 by (4.2). First, we consider the case $a = 0$. Because of (4.8) we have to compute the Gaussian area of a wedge of $\mathbb{R}^2$ with opening angle $\arccos(\rho)$. Hence, $\Pr_\rho(X_1 \le$

$0, Y_1 \geq 0) = \arccos(\rho)/(2\pi)$ by the rotational invariance of the Gaussian measure. As $\gamma^1(0) = 1/\sqrt{2\pi}$ we have equality in Lemma 4.4 for all $\rho \in (0,1)$.

For $a \neq 0$ we can, without loss of generality, assume that $a > 0$. Otherwise, we replace a by $-a$, X_1 by $-Y_1$ and Y_1 by $-X_1$ and use the symmetry of the Gaussian distribution. Thus, by (4.8) we get

$$\begin{aligned} \Pr_\rho(X_1 \leq a, Y_1 \geq a) &= \int_{\mathbb{R}^2} \mathbb{1}_{\{x\leq a\}} \mathbb{1}_{\{\rho x + \sqrt{1-\rho^2} y \geq a\}} \mathrm{d}\gamma^1(x)\mathrm{d}\gamma^1(y) \\ &= \frac{1}{2\pi} \int_0^\infty r e^{-r^2/2} \int_0^{2\pi} \mathbb{1}_{\{r\sin(\theta)\leq a\}} \mathbb{1}_{\{r\sin(\theta+\alpha)\geq a\}} \mathrm{d}\theta \mathrm{d}r \end{aligned}$$

where in the second step we transformed the integral to polar coordinates and used $\alpha = \arccos(\rho)$, i.e. $\rho\sin(\theta) + \sqrt{1-\rho^2}\cos(\theta) = \cos(\alpha)\sin(\theta) + \sin(\alpha)\cos(\theta) = \sin(\theta+\alpha)$.

Next, we evaluate the inner integral. If $a \geq r$ then $\{r\sin(\theta+\alpha) \geq a\}$ is empty and thus the inner integral is zero. If $a < r$ then $I_r := \{\theta \in [0, 2\pi]; r\sin(\theta) \geq a\}$ is an interval and the inner integral equals the size of $(I_r - \alpha) \setminus I_r$. If $I_r - \alpha$ and I_r intersect then the size is equal to α. If they do not intersect then the size is $\pi - 2\arcsin(a/r)$. Since the intersection is empty if and only if $\alpha \geq \pi - 2\arcsin(a/r)$ we get

$$\begin{aligned} \int_0^{2\pi} \mathbb{1}_{\{r\sin(\theta)\leq a\}} \mathbb{1}_{\{r\sin(\theta+\alpha)\geq a\}} \mathrm{d}\theta &= \mathbb{1}_{\{a<r\}} \min\left\{\alpha, \pi - 2\arcsin\left(\frac{a}{r}\right)\right\} \\ &= \begin{cases} \mathbb{1}_{\{a<r\}}\alpha, & r \geq a/\cos(\alpha/2), \\ \mathbb{1}_{\{a<r\}}(\pi - 2\arcsin(a/r)), & r < a/\cos(\alpha/2), \end{cases} \end{aligned}$$

where we used that $\alpha \leq \pi - 2\arcsin(a/r)$ is equivalent to $r \geq a/\cos(\alpha/2)$ since arcsin is monotonically increasing.

By integrating with respect to r we get

$$\begin{aligned} 2\pi \Pr_\rho(X_1 \leq a, Y_1 \geq a) &= \int_a^\infty \alpha r e^{-r^2/2} \mathrm{d}r - \int_a^{a/\cos(\alpha/2)} r e^{-r^2/2} (\alpha - \pi + 2\arcsin(a/r)) \mathrm{d}r \\ &= \alpha e^{-a^2/2} - \int_a^{a/\cos(\alpha/2)} r e^{-r^2/2} (\alpha - \pi + 2\arcsin(a/r)) \mathrm{d}r. \end{aligned}$$

For $\rho \to 1$ we have $\alpha = \arccos(\rho) \to 0$ and thus $\cos(\alpha/2) \sim 1 - \alpha^2/8$, i.e. $a/\cos(\alpha/2) \sim a + a\alpha^2/8$. The integration of above integral runs over an interval of length of order α^2. Since the integrand is bounded the integral divided by α goes to zero as α goes to zero. Hence, the assertion follows. □

Note that

$$\gamma^+(\{x \in \mathbb{R}^n; x_1 \leq a\}) = \liminf_{r\to 0} \int_a^{a+r} \frac{e^{-t^2/2}}{\sqrt{2\pi} r} \mathrm{d}t = \liminf_{r \to 0} \int_0^1 \frac{e^{-(rs-a)^2/2}}{\sqrt{2\pi}} \mathrm{d}s = \gamma^1(a) \tag{4.9}$$

and the previous lemma show that the inequality in Corollary 4.3 is sharp in the limit $\rho \to 1$ for half-spaces.

Now, Theorem 1.3 is a simple consequence of Theorem 1.4, more precisely (1.4), and Lemma 4.4.

Proof of Theorem 1.3. Let $A \subset \mathbb{R}^n$ be a measurable set and $B := \{x \in \mathbb{R}^n; x_1 \leq \Phi^{-1}(\gamma(A))\}$. Then B is a half-space with $\gamma(A) = \gamma(B)$. Taking the limit $\rho \uparrow 1$ in Corollary 4.3 yields

$$\limsup_{\rho\uparrow 1} \frac{\sqrt{2\pi}\Pr_\rho(X \in A, Y \notin A)}{\arccos(\rho)} \leq \gamma^+(A).$$

Using (1.4) for $0 < \rho < 1$ we get

$$\limsup_{\rho\uparrow 1} \frac{\sqrt{2\pi}\Pr_\rho(X \in B, Y \notin B)}{\arccos(\rho)} \leq \limsup_{\rho\uparrow 1} \frac{\sqrt{2\pi}\Pr_\rho(X \in A, Y \notin A)}{\arccos(\rho)} \leq \gamma^+(A).$$

Using Lemma 4.4 and (4.9) we get that the left-hand side equals $\gamma^1(\Phi^{-1}(\gamma(A))) = I(\gamma(A))$. This concludes the proof of Theorem 1.3. □

A. Auxiliary results

In this section, we collect some auxiliary results which are either of a technical nature or well-known abstract results. We start with a helpful lemma for computing the conditional expectation which we used in the proofs of Theorem 1.4 and Proposition 4.2.

Lemma A.1. *Let X and Y be independent random variables and φ a measurable map such that $\mathbb{E}|\varphi(X,Y)| < \infty$. If we define $g(x) := \mathbb{E}\varphi(x,Y)$ then $\mathbb{E}[\varphi(X,Y)|X] = g(X)$.*

The proof can be found in Example 5.1.5 on page 225 in [3]. The rest of the section provides tools for the proof of Theorem 1.5 beginning with the computation of an Itô differential.

Lemma A.2. *For every $x \in \mathbb{R}^n$ the process*

$$F_t(x) := \gamma_{W_t,\sqrt{1-t}}(x)$$

is a local martingale satisfying the stochastic differential equation

$$F_0(x) = \gamma(x), \quad dF_t(x) = (1-t)^{-1} F_t(x) \langle x - W_t, dW_t \rangle. \tag{A.1}$$

For any measurable function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ with $\phi(x) < C_1 + C_2|x|^p$ for some constants $C_1, C_2, p > 0$, the process

$$t \mapsto \int_{\mathbb{R}^n} \phi(x) F_t(x)\, dx$$

is a martingale with respect to the filtration $(\mathcal{F}_t)_t$ with

$$d\int_{\mathbb{R}^n} \phi(x) F_t(x)\, dx = (1-t)^{-1} \left\langle \int_{\mathbb{R}^n} \phi(x)(x - W_t) F_t(x)\, dx, dW_t \right\rangle \tag{A.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^n$.

Proof. We set

$$g_{x,t}(y) := \gamma_{y,\sqrt{1-t}}(x) = \frac{1}{(2\pi(1-t))^{n/2}} \exp\left(-\frac{|x-y|^2}{2(1-t)}\right). \tag{A.3}$$

Direct calculations show

$$\nabla_y g_{x,t}(y) = \frac{x-y}{1-t} g_{x,t}(y), \quad \Delta_y g_{x,t}(y) = \left(\frac{|x-y|^2}{(1-t)^2} - \frac{n}{1-t}\right) g_{x,t}(y) \tag{A.4}$$

and

$$\frac{\partial}{\partial t} g_{x,t}(y) = \left(\frac{n}{2(1-t)} - \frac{|x-y|^2}{2(1-t)^2}\right) g_{x,t}(y).$$

Thus, Itô's formula yields

$$\begin{aligned} \mathrm{d}F_t(x) = \mathrm{d}g_{x,t}(W_t) &= \frac{\partial}{\partial t} g_{x,t}(W_t)\mathrm{d}t + \langle \nabla g_{x,t}(W_t), \mathrm{d}W_t\rangle + \frac{1}{2}\Delta g_{x,t}(W_t)\mathrm{d}t \\ &= \langle \nabla g_{x,t}(W_t), \mathrm{d}W_t\rangle = (1-t)^{-1} F_t(x)\langle x - W_t, \mathrm{d}W_t\rangle \end{aligned}$$

which proves (A.1) and establishes that $F_t(x)$ is a local martingale. (Compare Exercise 7.12 (d) in [11].)

Let $\phi\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function with $|\phi(x)| \le C_1 + C_2|x|^p$ for all $x \in \mathbb{R}^n$ and some constants $C_1, C_2, p > 0$. Then we are allowed to interchange differentiation and integration which yields

$$\nabla \int_{\mathbb{R}^n} \phi(x) g_{x,t}(y)\mathrm{d}x = \int_{\mathbb{R}^n} \phi(x) \nabla g_{x,t}(y)\mathrm{d}x, \quad \left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right) \int_{\mathbb{R}^n} \phi(x) g_{x,t}(y)\mathrm{d}x = 0$$

where we used (A.3) and (A.4). Applying Itô's formula implies (A.2).

Moreover, as

$$\int_{\mathbb{R}^n} \phi(x) F_t(x)\mathrm{d}x = \mathbb{E}\left[\phi(W_1)|\mathcal{F}_t\right]$$

the left hand side defines a martingale. ☐

In the above proof of Theorem 1.5 we used the following version of the Dambis/Dubins-Schwarz Theorem which is proved in Theorem V. (1.7) in [13]. A probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$ with a filtration $(\tilde{\mathcal{F}}_t)_t$ and a surjective map $\pi\colon \tilde{\Omega} \to \Omega$ is called an *enlargement* of the probability space $(\Omega, \mathcal{A}, P)$ with filtration $(\mathcal{F}_t)_t$ if $\pi^{-1}(\mathcal{F}_t) \subset \tilde{\mathcal{F}}_t$ for all t and $\pi_\#(\tilde{P}) = P$. We extend a process X defined on Ω to $\tilde{\Omega}$ by setting $X(\tilde{\omega}) = X(\omega)$ if $\pi(\tilde{\omega}) = \omega$.

Theorem A.3. *Let M be a continuous local martingale with respect to the filtration $(\mathcal{F}_t)_t$ which vanishes at zero. We denote the inverse function of the quadratic variation $[M]_t$ by T_t. Then there exist an enlargement $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$ with a filtration $(\tilde{\mathcal{F}}_t)_t$ and a Brownian motion β on $\tilde{\Omega}$ independent of M such that*

$$B_t = \begin{cases} M_{T_t}, & \text{if } t < [M]_\infty \\ M_\infty + \beta_{t-[M]_\infty}, & \text{if } t \geq [M]_\infty \end{cases}$$

is a standard Brownian motion.

The remaining part of this section is based on chapter 5 in [5]. We start with Definition 5.5 in [5] which gives an unusual definition of a regular measure space.

Definition A.4. *A measure space $(X, \mathscr{B}, \mu)$ is called* regular *if X is a compact metric space and $\mathscr{B}$ consists of all Borel sets in X.*

A measurable map $\phi\colon (X, \mathscr{B}, \mu) \to (Y, \mathscr{D}, \nu)$ between two measure spaces $(X, \mathscr{B}, \mu)$ and $(Y, \mathscr{D}, \nu)$ is called *measure preserving* if ν is the pushforward measure of μ under ϕ, i.e. $\phi_\#\mu = \nu$ or $\nu(A) = \mu(\phi^{-1}(A))$ for all $A \in \mathscr{D}$.

If $\phi\colon (X, \mathscr{B}, \mu) \to (Y, \mathscr{D}, \nu)$ is a measure preserving map then the map

$$\Phi\colon L^2(Y, \mathscr{D}, \nu) \to L^2(X, \mathscr{B}, \mu), \quad f \mapsto f \circ \phi$$

defines an isometry by the change of variables formula. In particular, the image $\operatorname{Ran}\Phi \subset L^2(X, \mathscr{B}, \mu)$ is a closed linear subspace. We denote the orthogonal projection onto this subspace by P. Moreover, for every $f \in L^2(X, \mathscr{B}, \mu)$ there exists a unique $g \in L^2(Y, \mathscr{D}, \nu)$ such that $\Phi(g) = Pf$. This function g is called *conditional expectation* and denoted by $\mathbb{E}[f|\mathbf{Y}]$. Note that the measure preserving map ϕ is suppressed in this notation. If $Y = X$, $\phi = \mathrm{id}_X$, $\mu = \nu$ and $\mathscr{D} \subset \mathscr{B}$ a sub-σ-algebra then this definition of the conditional expectation coincides with the usual definition.

The next result is a special case of Theorem 5.8 on page 108 in [5] where we consider only measure preserving maps instead of more general homomorphisms of measure spaces.

Theorem A.5 (Disintegration)**.** *Let $(X, \mathscr{B}, \mu)$ is a regular probability space in the sense of Definition A.4, $(Y, \mathscr{D}, \nu)$ a measure space and $\phi\colon X \to Y$ a measure preserving map. Then there exists a measurable map from Y to $\mathscr{M}(X)$, denoted by $y \mapsto \mu_y$, satisfying*

(i) For every $f \in L^1(X, \mathscr{B}, \mu)$ we have $f \in L^1(X, \mathscr{B}, \mu_y)$ for almost every $y \in Y$ and $\mathbb{E}[f|\mathbf{Y}](y) = \int f d\mu_y$ for almost every $y \in Y$.

(ii) For every $f \in L^1(X, \mathscr{B}, \mu)$ we have

$$\int_Y \left(\int_X f d\mu_y \right) d\nu(y) = \int_X f d\mu.$$

Let $(X_1, \mathscr{B}_1, \mu_1)$ and $(X_2, \mathscr{B}_2, \mu_2)$ be two regular probability spaces in the sense of Definition A.4 with measure preserving maps

$$\phi_1\colon (X_1, \mathscr{B}_1, \mu_1) \to (Y, \mathscr{D}, \nu), \qquad \phi_2\colon (X_2, \mathscr{B}_2, \mu_2) \to (Y, \mathscr{D}, \nu).$$

Applying Theorem A.5, we get the two disintegrations $y \mapsto \mu_{1,y}$ and $y \mapsto \mu_{2,y}$. Thus, we can define

$$P(A) = \int_Y \mu_{1,y} \times \mu_{2,y}(A) \mathrm{d}\nu(y) \tag{A.5}$$

on the measurable space $(X_1 \times X_2, \mathscr{B}_1 \times \mathscr{B}_2)$. Using the monotone convergence theorem it is easy to see that P defines a measure on $(X_1 \times X_2, \mathscr{B}_1 \times \mathscr{B}_2)$ which is a probability measure as

$$P(X_1 \times X_2) = \int_Y \mathrm{d}\nu = 1.$$

Moreover, we have the following lemma.

Lemma A.6. *The measures μ_1 and μ_2 are the pushforward measures of P under π_1 and π_2, respectively.*

Proof. For $A_1 \in \mathscr{B}_1$ we have

$$\begin{aligned}(\pi_1)_\# P(A_1) =& P(\pi_1^{-1}(A_1)) \\ =& \int_Y \mu_{1,y} \times \mu_{2,y}(A_1 \times X_2) \mathrm{d}\nu(y) \\ =& \int_Y \int_X \mathbb{1}_{A_1} \mathrm{d}\mu_{1,y} \mathrm{d}\nu(y) \\ =& \int_X \mathbb{1}_{A_1} \mathrm{d}\mu_1 \\ =& \mu_1(A_1).\end{aligned}$$

Here, we used property (ii) of Theorem A.5 in the fourth step. This shows $(\pi_1)_\# P = P_1$. Similarly, $(\pi_2)_\# P = P_2$ is proved. ☐

Introducing the canonical projections $\pi_1 \colon X_1 \times X_2 \to X_1$ and $\pi_2 \colon X_1 \times X_2 \to X_2$ we have the following situation

$$\begin{array}{ccc} & (X_1 \times X_2, \mathscr{B}_1 \times \mathscr{B}_2, P) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ (X_1, \mathscr{B}_1, \mu_1) & & (X_2, \mathscr{B}_2, \mu_2) \\ {}_{\phi_1}\searrow & & \swarrow_{\phi_2} \\ & (Y, \mathscr{D}, \nu) & \end{array}$$

The following Proposition is proved as Proposition 5.11 on page 112 in [5].

Proposition A.7. *The above diagram is commutative, i.e. $\phi_1 \circ \pi_1 = \phi_2 \circ \pi_2$.*

Combining the previous results and observing that a Brownian motion defines a homomorphism to the space of real-valued continuous functions with an appropriate σ-algebra we get

Theorem A.8. *Let $(\Omega_1, \Sigma_1, P_1)$ and $(\Omega_2, \Sigma_2, P_2)$ be two regular probability spaces in the sense of Definition A.4 and $B_1 \colon \Omega_1 \times [0, \infty) \to \mathbb{R}$, $B_2 \colon \Omega_2 \times [0, \infty) \to \mathbb{R}$ be two standard Brownian motions over the probability spaces Ω_1, Ω_2 respectively. Then there exists a probability space (Ω, Σ, P) and two measurable functions $\pi_1 \colon \Omega \to \Omega_1$ and $\pi_2 \colon \Omega \to \Omega_2$ such that*

(i) P_1 is the pushforward measure of P under π_1, i.e. $(\pi_1)_\# P = P_1$,

(ii) P_2 is the pushforward measure of P under π_2, i.e. $(\pi_2)_\# P = P_2$,

(iii) For P-almost every $\omega \in \Omega$ one has $B_1(\pi_1(\omega), t) = B_2(\pi_2(\omega), t)$ for all $t \in [0, \infty)$.

The proof follows the proof of Theorem 10 in [4].

Proof. The goal is to define an appropriate measure space $(Y, \mathscr{D}, \nu)$ such that the Brownian motions B_1 and B_2 induce measure preserving maps $(\Omega_1, \Sigma_1, P_1) \to (Y, \mathscr{D}, \nu)$ and $(\Omega_2, \Sigma_2, P_2) \to (Y, \mathscr{D}, \nu)$ and apply Theorem A.5.

It is natural to take $Y = C([0, \infty), \mathbb{R})$, the space of continuous function from $[0, \infty)$ to $\mathbb{R}$, with the σ-algebra $\mathscr{D}$ generated by the sets

$$\{\omega \in C([0, \infty), \mathbb{R}); k \in \mathbb{N}, F_1, \ldots, F_k \in \mathscr{B}(\mathbb{R}), 0 < t_1 < \ldots < t_k, \omega(t_1) \in F_1, \ldots, \omega(t_k) \in F_k\} \tag{A.6}$$

and the unique probability measure ν which fulfills

$$\nu(\delta_{t_1} \in F_1, \ldots, \delta_{t_k} \in F_k) = \int_{F_1 \times \ldots \times F_k} \gamma^1_{0, \sqrt{t_1}}(x_1) \ldots \gamma^1_{x_{k-1}, \sqrt{t_k - t_{k-1}}}(x_k) \mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{A.7}$$

for $0 < t_1 < \ldots < t_k$, $F_1, \ldots, F_k \in \mathscr{B}(\mathbb{R})$ and $\delta_t \colon C([0, \infty), \mathbb{R}) \to \mathbb{R}, \omega \mapsto \omega(t)$.

For $i = 1, 2$ consider the maps

$$\phi_i \colon (\Omega_i, \Sigma_i, P_i) \to (Y, \mathscr{D}, \nu), \quad \omega \mapsto B_i(\cdot)(\omega).$$

Clearly, $B_i(\cdot)(\omega) \in Y$ for all $\omega \in \Omega_i$. Moreover, let A be a set as in (A.6). Then

$$\phi_i^{-1}(A) = \{B_i(t_1) \in F_1, \dots, B_i(t_k) \in F_k\},$$

i.e. ϕ_i is measurable. Thus, $(\phi_i)_\# P_i$ and ν define two probability measures on Y. By properties of the Brownian motion and (A.7) they coincide on the sets in (A.6) and therefore ϕ_i is a measure preserving map for $i = 1, 2$ by an easy application of Dynkin's Theorem.

Thus, we can apply above considerations and define $\Omega := \Omega_1 \times \Omega_2$, $\Sigma := \Sigma_1 \times \Sigma_2$ and P as in (A.5). Moreover, let π_1 and π_2 be the canonical projections onto the first and second component of Ω, respectively. Then Lemma A.6 implies (i) and (ii) of the Theorem and Proposition A.7 yields (iii). This concludes the proof of Theorem A.8. □

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