Handout: Introduction to the Free Probability

Classical probability

Cumulants

Def: For a random variable X with $m_k = \mathbb{E}[X^k]$ define the *moment-generating function*:

$$G(t) = \mathbb{E}[e^{tX}] = 1 + \sum_{k=1}^{\infty} m_k \frac{t^k}{k!},$$

and the *cumulant-generating function*

$$g(t) = \log G(t) = 1 + \sum_{k=1}^{\infty} c_k \frac{t^k}{k!}.$$

The coefficients c_k are called the *cumulants*.

Corollary: If X and Y are independent, then $c_n(X+Y) = c_n(X) + c_n(Y)$.

Example: For the standard Gaussian random variable we have $c_i = \delta_{i2}$ for all $i \geqslant 1$.

Theorem (G.-C. Rota, 1964): Denote by $P(n)$ the set of all partitions of $\{1, 2, \ldots, n\}$. Then there is a combinatorial description of cumulants:

$$m_n = \sum_{\pi \in P(n)} \prod_{B \in \pi} c_{|B|}.$$

Example:

$$\begin{aligned}
c_1(X) &= m_1(X) = \mathbb{E}X \\
c_2(X) &= m_2(X) - m_1(X)^2 = \mathbb{D}X \\
c_3(X) &= m_3(X) - 3m_2(X)m_1(X) + 2m_1(X)^3 \\
c_4(X) &= m_4(X) - 4m_3(X)m_1(X) - 3m_2(X)^2 + 12m_2(X)m_1(X)^2 - 6m_1(X)^4.
\end{aligned}$$

Def: For random variables $X_1, \ldots, X_n$ define the *joint moment-generating function*:

$$G(\mathbf{t}) = \mathbb{E}[e^{t_1 X_1 + \ldots + t_n X_n}] = \sum_{\text{multi-index } \alpha} m_\alpha \frac{t^\alpha}{\alpha!},$$

and the *cumulant-generating function*

$$g(\mathbf{t}) = \log G(\mathbf{t}) = \sum_{\text{multi-index } \alpha} c_\alpha \frac{t^\alpha}{\alpha!}.$$

The coefficients $m_{(1_1, \ldots, 1_n)}$ and $c_{(1_1, \ldots, 1_n)}$ are called the *mixed moments* m_n and *cumulants* c_n.

Theorem:

$$m_n(X_1, \ldots, X_n) = \mathbb{E}[X_1 \ldots X_n].$$
$$m_n(X_1, \ldots, X_n) = \sum_{\pi \in P(n)} \prod_{B \in \pi} c_{|B|}(X_i \colon i \in B).$$

Theorem: If X and Y are independent random variables, then their joint cumulants vanish:

$$c_k(X, X, \ldots, Y, Y) = 0.$$

Conversly, if the joint cumulants vanish, then X and Y are *subindependent*, i.e., for any polynomials p and q we have $\mathbb{E}[p(X)q(Y)] = \mathbb{E}[p(X)]\mathbb{E}[q(Y)]$.

Free probability

Non-crossing partitions

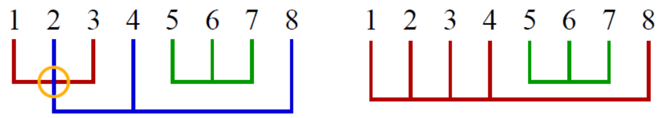

Figure 1: Crossing (left) and non-crossing (right) partitions of $\{1, \ldots, 8\}$

Def: Denote by NC(n) the set of all non-crossing partitions of $\{1, 2, \ldots, n\}$ (see Fig. 1).

Free cumulants

Def: For a random variable X with $m_k = \mathbb{E}[X^k]$ define the *non-crossing* or *free cumulants* κ_i:

$$m_n = \sum_{\pi \in \mathsf{NC}(n)} \prod_{B \in \pi} \kappa_{|B|}.$$

Similary to the classical case define the *joint free cumulants*.

Example: Free "Gaussian", i.e., random variable with $\kappa_i = \delta_{i2}$, is the Wigner semicircle variable with the density:

$$\rho(t) = \frac{1}{2\pi} \sqrt{4 - t^2}\, \mathbb{1}_{|t| \leqslant 2}.$$

Note: $\kappa_1 = c_1$, $\kappa_2 = c_2$ and $\kappa_3 = c_3$.

Non-commutative probability space

Def: A pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a complex unital algebra and τ a unital linear functional is a *non-commutative probability space*. An element $X \in \mathcal{A}$ is a *free random variable* and $m_n = \tau[X^n]$ are its *moments*.

Example: Classical probability: $\mathcal{A} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $\tau = \mathbb{E}$.

Free independence

Def (D.-V. Voiculescu, 1987): Free random variables X and Y are *freely independent* if

$$\tau[f_1(X) g_1(Y) \ldots f_k(X) g_k(Y)] = 0$$

whenever f_i and g_i are polynomials such that $\tau[f_i(X)] = \tau[g_j(Y)] = 0$.

Theorem (R. Speicher, 1997): It is equivalent to the vanishing of the joint free cumulants.

Voiculescu's algorithm

Def: For a real measure μ define the *Stieltjes transform*:

$$G_X(z) = \int_{\mathbb{R}} \frac{1}{z - t} \mu(\mathrm{d}t).$$

Proposition: The Stieltjes transform stores the moments of the measure μ in the following way:

$$G(z) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{\int t^n \mu(\mathrm{d}t)}{z^n} = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}.$$

Voiculescu's algorithm is used to find the moments of the sum of freely independent random variables X and Y. We assume that there are compactly supported real distributions μ_X and μ_Y with the same moments as X and Y. Then the algorithm is the following:

Input: μ_X and μ_Y.

Step 1. Compute the Stieltjes transforms of X and Y:

$$G_X(z) = \int\limits_{\mathbb{R}} \frac{1}{z-t}\mu_X(\mathrm{d}t),\ G_Y(z) = \int\limits_{\mathbb{R}} \frac{1}{z-t}\mu_Y(\mathrm{d}t)$$

Step 2. Solve the first Voiculescu functional equations to get *Voiculescu transforms* of X and Y:

$$(G_X \circ V_X)(w) = w, \quad (G_Y \circ V_Y)(w) = w$$

subject to $V_X(w) \sim \frac{1}{w}$ near $w = 0$.

Step 3. Remove the principal part to get *R-transforms*:

$$R_X(w) = V_X(w) - \frac{1}{w}, \quad R_Y(w) = V_Y(w) - \frac{1}{w},$$

add to get the R-transform of $X + Y$:

$$R_{X+Y}(w) := R_X(w) + R_Y(w),$$

restore principal part to get the Voiculescu transform of $X + Y$,

$$V_{X+Y}(w) := R_{X+Y}(w) + \frac{1}{w}.$$

Step 4. Solve the second Voiculescu functional equation,

$$(V_{X+Y} \circ G_{X+Y})(z) = z,$$

subject to $G_{X+Y}(z) \sim \frac{1}{z}$ near $z = \infty$.

Output: $G_{X+Y}(z)$ — Stieltjes transform of μ_{X+Y}, which encodes all information about the moments.

To obtain the measure μ_{X+Y} the inverse Stieltjes transform can be calculated:

$$\mu_{X+Y}(\mathrm{d}t) = -\frac{1}{\pi}\lim_{\varepsilon\to 0} \mathrm{Im} G_{X+Y}(t + i\varepsilon).$$

Application. Random walks on free groups

Theorem (P. Gerl and W. Woess, 1986): Random walk on the free group $\mathbb{F}_n = \langle a_1, \dots a_n \rangle = \mathbb{Z}^{*n}$ is transient, i.e., does not return to the starting point with positive probability, for $n \geqslant 2$.

References

[1] J. Novak, M. LaCroix, "Three lectures on free probability", arXiv:1205.2097, 2012.

[2] R. Speicher, "Free probability theory and noncrossing partitions", in 39 Séminaire Lotharingien de Combinatoire, Thurnau, Germany, 1997.

[3] T. Tao, "254A, Notes 5: Free Probability", http://terrytao.wordpress.com/2010/02/10/245a-notes-5-free-probability