Szemerédi's Theorem via Ergodic Theory

Rotation Project Final Presentation

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July 24, 2014

Historical Overview of Results

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Conjecture of P. Erdős & P. Turán (1936)**.** True reason: at least one of the color classes has positive **upper density**.

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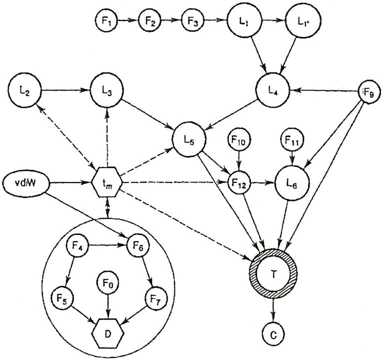
Szemerédi's Theorem

Roadmap of Szemerédi's Original Proof



The diagram represents an approximate flow chart for the accompanying proof of Szemerédi's theorem. The various symbols have the following meanings: F_k ≡ Fact k, L_k ≡ Lemma k, T ≡ Theorem, C ≡ Corollary, D ≡ Definitions of B, S, P, α, β, etc., t_m ≡ Definition of t_m, vdW ≡ van der Waerden's theorem, F_0 ≡ "If $f: R^+ \to R^+$ is subadditive then $\lim_{n\to\infty} \frac{f(n)}{n}$ exists".

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1. **Ergodic Theory.** Basic concepts, notation, ergodic theorems, ergodic decomposition.
2. **Correspondence Principle.** Converting problems in additive combinatorics into problems about dynamical systems.

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4. **Roth's Theorem** ($k = 3$)**.** Putting the pieces together.

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Remark. We mainly follow the exposition of the essay Szemerédi's Theorem via Ergodic Theory by **Yufei Zhao** (2011).

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$\Rightarrow$ **Topological Dynamics**

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Proposition. If a system X is ergodic, then $\mathbb{E}(f|\mathcal{X}^T) = \mathbb{E}(f)$, where $\mathbb{E}(f) = \int_X f \,\mathrm{d}\mu$ is the usual expectation.

Ergodic Theory III. – Ergodic Theorems

"General Form" of Ergodic Theorems.

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$$\mathrm{Av}_N(T^n f) \longrightarrow \mathbb{E}(f|\mathcal{X}^T) \quad \text{in } L^2$$

as $N \to \infty$. If X is ergodic, then the limit equals $\mathbb{E}(f)$.

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$\Longrightarrow$ **We can assume ergodicity of systems in certain types of proofs** (including the proof of Szemerédi's Theorem).

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Idea. Working in an appropriate subspace of X, we shall turn it into a **measure space** via a T-invariant measure μ. $\rightsquigarrow$ **Goal.**

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Note. For $k = 2$, the claim is a trivial also in this setting. (Also compare with: **Poincaré Recurrence Theorem**.)

Correspondence Principle III. – Multiple Recurrence

Theorem (E. Szemerédi, 1975)

If $A \subseteq \mathbb{Z}$, such that $\overline{\delta}(A) > 0 \Longrightarrow A$ contains a k-term arithmetic progression for each $k \in \mathbb{N}_+$.

Correspondence $\Updownarrow$ **Principle**

Multiple Recurrence Theorem (H. Furstenberg, 1977)

Let $(X, \mathcal{X}, \mu, T)$ be a system, and $k \in \mathbb{N}_+$. Then for any $E \in \mathcal{X}$ with $\mu(E) > 0$ there exists some $n \in \mathbb{N}_+$ such that

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Existence. $\mu_N := \frac{1}{2N+1} \sum_{n=-N}^{N} \delta_{T^n a}$. **Homework:** The sequence $(\mu_N)_{N \in \mathbb{N}}$ has some T-invariant weak limit μ, for which $\mu(E) > 0$.

[Use the assumption $\overline{\delta}(A) > 0$ & the **Banach–Alaoglu Theorem**.]

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Multiple Recurrence Theorem (alternative form)

$(X, \mathcal{X}, \mu, T)$ is a system, and $k \in \mathbb{N}_+$. $\forall E \in \mathcal{X}$ with $\mu(E) > 0$,

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whenever $f \in L^\infty(X)$, $f \geq 0$, and $\mathbb{E}(f) > 0$. A system X is **SZ** if it is SZ of every level. **Ultimate Goal:** Every system is SZ.

Ideas of the Proof – Types of Systems

Measure Preserving Systems

(according to the behavior of T)

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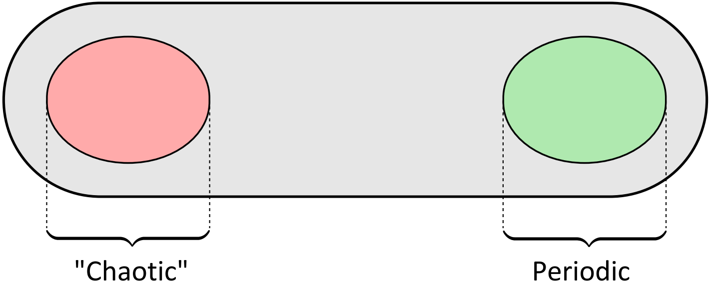

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Weak mixing and **Almost Periodic/Compact** systems.

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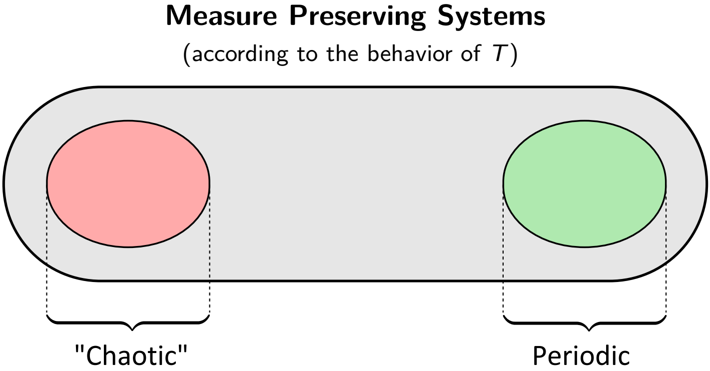

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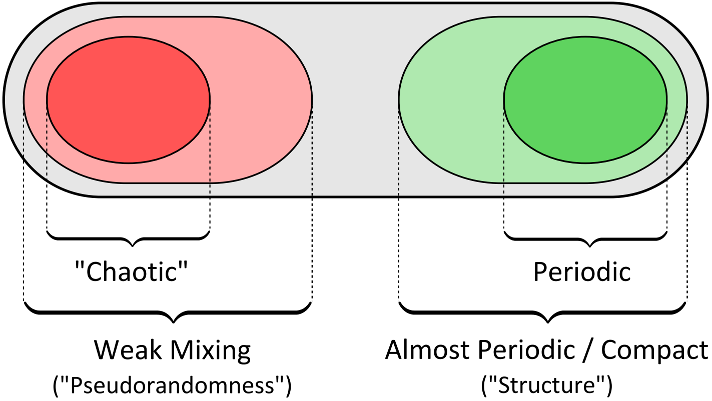

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Definition. $v \in V$, $(v_n)_{n\in\mathbb{N}} \subset V$ normed. **$\text{D-lim}_{n\to\infty} v_n = v$**, if for any $\varepsilon > 0$ we have $\overline{\delta}\left(\{n \in \mathbb{N} \mid \|v_n - v\| > \varepsilon\}\right) = 0$.

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Comparing weak mixing and ergodic systems:

Proposition. Weak mixing $\Longrightarrow$ ergodicity (but not vica versa).

Proposition. X w. m. $\Longleftrightarrow$ $X \times X$ w. m. $\Longleftrightarrow$ $X \times X$ ergodic.

Remark. X ergodic $\not\Longrightarrow$ $X \times X$ ergodic [irrational rotation of S^1].

Weak Mixing Functions

Definition. In a system $(X, \mathcal{X}, \mu, T)$ a function $f \in L^2(X)$ is called **weak mixing** if $\text{D-lim}_{n\to\infty} \langle T^n f, f\rangle = 0$.

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Characterization of w. m. systems by w. m. functions:

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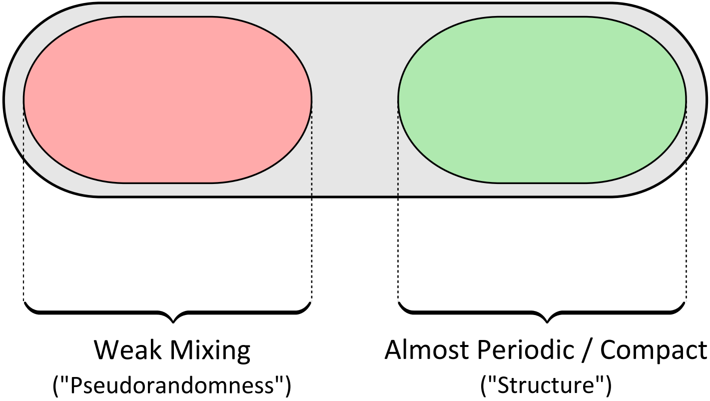
Theorem. Every weak mixing system is SZ.

Ideas of the Proof – Types of Systems

Measure Preserving Systems

(according to the behavior of T)



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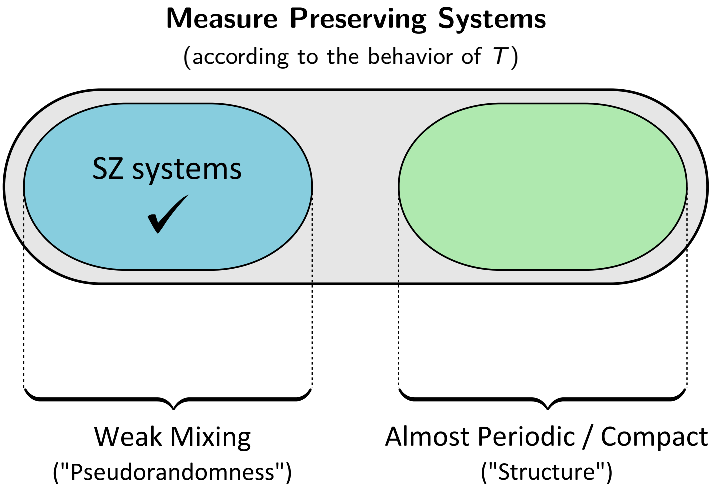

Compact Systems & Almost Periodic Functions

Definition. A function $f \in L^2(X)$ is **almost periodic** if for every $\varepsilon > 0$, the set $S_\varepsilon = \{n \in \mathbb{Z} \mid \|f - T^n f\|_2 < \varepsilon\}$ has bounded gaps, which means $\exists N > 0 \colon S_\varepsilon \cap [m, m+N] \neq \emptyset$ for all $m \in \mathbb{Z}$.

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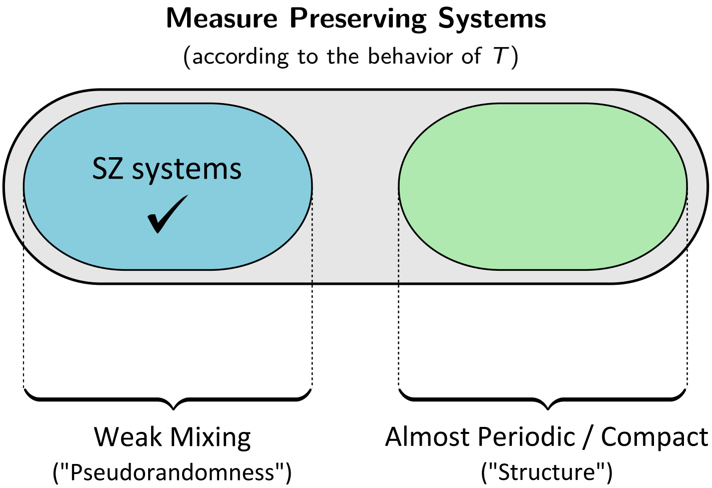

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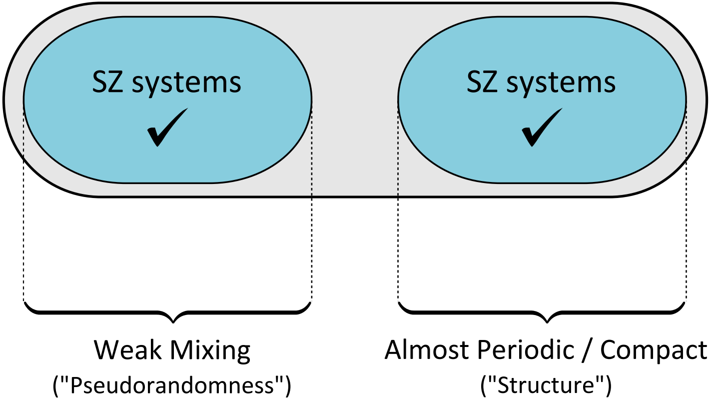

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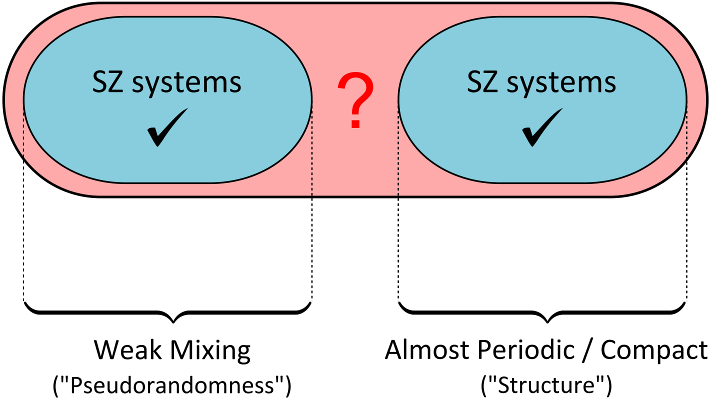

Weak Mixing & Almost Periodic Components

Notation. $WM(X) := \{f \in L^2(X) \mid f \text{ is weak mixing}\}$
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Message. Unless a system is completely "pseudorandom" ($\leftrightsquigarrow WM$), it must contain some "structured" ($\leftrightsquigarrow AP$) piece.

Now we are ready to prove **Roth's Theorem**.

Roth's Theorem – Statement of the Theorem

Theorem (Roth)**.** Every subset A of $\mathbb{Z}$ with $\overline{\delta}(A) > 0$ contains a 3-term arithmetic progression.

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Theorem (Roth). Every system is SZ of level 3. In other words, let $(X, \mathcal{X}, \mu, T)$ be a system. Then for every $f \in L^\infty(X)$ with $f \geq 0$ and $\mathbb{E}(f) > 0$, we have

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Intuition. We get rid of the weak mixing part of the system, and project it onto its almost periodic piece, which is known to be SZ, as it is a compact system.

Roth's Theorem – Key Ingredients

Proposition (♠). $L^2(X) = WM(X) \oplus AP(X)$ as an orthogonal direct sum of Hilbert spaces. Therefore each $f \in L^2(X)$ can be written as $f = f_{WM} + f_{AP} \in WM(X) \oplus AP(X)$.

Proposition (♣). Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system. Then for any $f, g \in L^\infty(X)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(T^n f \, T^{2n} g - T^n f_{AP} T^{2n} g_{AP} \right) = 0 \text{ in } L^2.$$

[The proof relies on **von Neumann's mean Ergodic Theorem**.]

Theorem (■). Every compact system is SZ.

Roth's Theorem – Proof

Let $f \in L^\infty(X)$ with $f \geq 0$ and $\mathbb{E}(f) > 0$.

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as $\mathbb{E}(f_{AP}) = \mathbb{E}(\mathbb{E}(f|\mathcal{X}_{AP})) = \mathbb{E}(f) > 0$ (see Appendix). □

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What About the Set of Primes?

Let $\mathcal{P} := \{\text{nonnegative primes}\} \subset \mathbb{Z}$, and $\pi(x) := \mathcal{P} \cap [0, x]$.

Claim (elementary). $\lim_{x\to\infty} \pi(x)/x = 0. \implies \boldsymbol{\overline{\delta}(\mathcal{P}) = 0}$.

Prime Number Theorem. $\pi(x)/x \sim \log(x)^{-1} \quad (\text{as } x \to \infty)$.

Question. Longest arithmetic progression containing only primes?

Theorem (B. Green and T. Tao, 2004)

For any $k \in \mathbb{N}_+$ there exists a k-term arithmetic progression in $\mathcal{P}$.

Szemerédi's Theorem is a key ingredient of the proof.

Example (B. Perichon, J. Wróblewski, and G. Reynolds, 2010)

$43,142,746,595,714,191 + 23,681,770 \cdot 223,092,870 \cdot n$, for $n = 0$ to 25. $\implies$ 26-term arithmetic progression of primes.

Ergodic Decomposition: An Example

Goal. Decompose an arbitrary system into **ergodic components**.

Why? Ergodic theorems have **simple** forms for ergodic systems.

Example (a special finite case, but works in general)

$X = \{1,2,3,4,5,6\}$, $\mathcal{X} = 2^X$, μ: uniform, $T = (2\,3)(4\,5\,6) \in S_X$.

$\Longrightarrow T(1) = 1, T(2) = 3, T(3) = 2, T(4) = 5, T(5) = 6, T(6) = 4.$

NOT ergodic: $E = \{1\}$ is T-invariant, but $\mu(E) = 1/6 \notin \{0,1\}$.

BUT consider: $\mu_1 = \mathbb{1}_{\{1\}}$, $\mu_2 = (1/2) \cdot \mathbb{1}_{\{2,3\}}$, $\mu_3 = (1/3) \cdot \mathbb{1}_{\{4,5,6\}}$.

Note: $(X, \mathcal{X}, \mu_i, T)$ **IS** ergodic for $i = 1,2,3$. Moreover, we have

$$\boxed{\mu = \frac{1}{6}\mu_1 + \frac{1}{3}\mu_2 + \frac{1}{2}\mu_3} \Longrightarrow$$

weighted average of ergodic measures.

Szemerédi $\Longrightarrow$ Furstenberg

Lemma. Let $(X, \mathcal{X}, \mu, T)$ be a system, and $E \in \mathcal{X}$ with $\mu(E) > 0$.
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$\implies \exists F \in \mathcal{X}$, $\mu(F) > 0$, s. t. $\forall x \in F$: $\overline{\delta}\left(\{n \in \mathbb{Z} \mid T^n x \in E\}\right) > 0$.

Fix a system $(X, \mathcal{X}, \mu, T)$, $E \in \mathcal{X}$ with $\mu(E) > 0$, and $k \in \mathbb{N}_+$.

- $\mathcal{AP}_k := \{\boldsymbol{a} = (a_1, \ldots, a_k) \subset \mathbb{Z} \mid \boldsymbol{a}$ is arithmetic progression$\}$.
- $B_{\boldsymbol{a}} := \{x \in X \mid T^{a_i} x \in E,\ 1 \le i \le k\}$; $B_k := \bigcup_{\boldsymbol{a} \in \mathcal{AP}_k} B_{\boldsymbol{a}}$.

Note: $B_k = \left\{x \in X \mid \{n \in \mathbb{Z} \mid T^n x \in E\}\text{ contains some } \boldsymbol{a} \in \mathcal{AP}_k\right\}$.

Choose $F \in \mathcal{X}$ as in the **Lemma**. By **Szemerédi**, $\{n \in \mathbb{Z} \mid T^n x \in E\}$ contains some $\boldsymbol{a} \in \mathcal{AP}_k$ for each $x \in F$. $\implies F \subseteq B_k = \bigcup_{\boldsymbol{a} \in \mathcal{AP}_k} B_{\boldsymbol{a}}$, a countable union. $\implies$ **Since** $\mu(F) > 0$, $\exists \boldsymbol{b} \in \mathcal{AP}_k$: $\mu(B_{\boldsymbol{b}}) > 0$.

Szemerédi $\Longrightarrow$ Furstenberg

Lemma. Let $(X, \mathcal{X}, \mu, T)$ be a system, and $E \in \mathcal{X}$ with $\mu(E) > 0$.
$\Longrightarrow \exists F \in \mathcal{X}$, $\mu(F) > 0$, s. t. $\forall x \in F$: $\overline{\delta}\left(\{n \in \mathbb{Z} \mid T^n x \in E\}\right) > 0$.

Fix a system $(X, \mathcal{X}, \mu, T)$, $E \in \mathcal{X}$ with $\mu(E) > 0$, and $k \in \mathbb{N}_+$.

- $\mathcal{AP}_k := \{\boldsymbol{a} = (a_1, \ldots, a_k) \subset \mathbb{Z} \mid \boldsymbol{a}$ is arithmetic progression$\}$.
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Note: $B_k = \left\{x \in X \mid \{n \in \mathbb{Z} \mid T^n x \in E\}\right.$ contains some $\left.\boldsymbol{a} \in \mathcal{AP}_k\right\}$.

Choose $F \in \mathcal{X}$ as in the **Lemma**. By **Szemerédi**, $\{n \in \mathbb{Z} \mid T^n x \in E\}$ contains some $\boldsymbol{a} \in \mathcal{AP}_k$ for each $x \in F$. $\Longrightarrow F \subseteq B_k = \bigcup_{\boldsymbol{a} \in \mathcal{AP}_k} B_{\boldsymbol{a}}$, a countable union. $\Longrightarrow$ **Since** $\mu(F) > 0$, $\exists \boldsymbol{b} \in \mathcal{AP}_k$: $\mu(B_{\boldsymbol{b}}) > 0$.
$\Longrightarrow T^c B_{\boldsymbol{b}} \subseteq E \cap T^n E \cap \cdots \cap T^{(k-1)n} E$ for some $c \in \mathbb{Z}$ and $n \in \mathbb{N}_+$.

Szemerédi $\Longrightarrow$ Furstenberg

Lemma. Let $(X, \mathcal{X}, \mu, T)$ be a system, and $E \in \mathcal{X}$ with $\mu(E) > 0$.
$\Longrightarrow \exists F \in \mathcal{X},\ \mu(F) > 0$, s. t. $\forall x \in F\colon \overline{\delta}\left(\{n \in \mathbb{Z} \mid T^n x \in E\}\right) > 0$.

Fix a system $(X, \mathcal{X}, \mu, T)$, $E \in \mathcal{X}$ with $\mu(E) > 0$, and $k \in \mathbb{N}_+$.

- $\mathcal{AP}_k := \{\boldsymbol{a} = (a_1, \dots, a_k) \subset \mathbb{Z} \mid \boldsymbol{a}$ is arithmetic progression$\}$.
- $B_{\boldsymbol{a}} := \{x \in X \mid T^{a_i} x \in E,\ 1 \le i \le k\}$; $B_k := \bigcup_{\boldsymbol{a} \in \mathcal{AP}_k} B_{\boldsymbol{a}}$.

Note: $B_k = \left\{x \in X \mid \{n \in \mathbb{Z} \mid T^n x \in E\}\ \text{contains some}\ \boldsymbol{a} \in \mathcal{AP}_k\right\}$.

Choose $F \in \mathcal{X}$ as in the **Lemma**. By **Szemerédi**, $\{n \in \mathbb{Z} \mid T^n x \in E\}$ contains some $\boldsymbol{a} \in \mathcal{AP}_k$ for each $x \in F$. $\Longrightarrow F \subseteq B_k = \bigcup_{\boldsymbol{a} \in \mathcal{AP}_k} B_{\boldsymbol{a}}$, a countable union. $\Longrightarrow$ **Since** $\mu(F) > 0$, $\exists \boldsymbol{b} \in \mathcal{AP}_k\colon \mu(B_{\boldsymbol{b}}) > 0$.
$\Longrightarrow T^c B_{\boldsymbol{b}} \subseteq E \cap T^n E \cap \dots \cap T^{(k-1)n} E$ for some $c \in \mathbb{Z}$ and $n \in \mathbb{N}_+$.
$\Longrightarrow \mu(E \cap T^n E \cap \dots \cap T^{(k-1)n} E) \ge \mu(T^c B_{\boldsymbol{b}}) = \mu(B_{\boldsymbol{b}}) > 0$. □

Compact Factors

Definition. $\mathcal{Y}$ is ***T*-invariant**, if $TE, T^{-1}E \in \mathcal{Y}$ for any $E \in \mathcal{Y}$.

Definition. If $X = (X, \mathcal{X}, \mu, T)$ is a system, $X' = (X, \mathcal{X}', \mu, T)$ is called a **factor** of X if $\mathcal{X}'$ is a ***T*-invariant** sub-σ-algebra of $\mathcal{X}$.

A factor X' is **trivial**, if $\mu(E) \in \{0, 1\}$ for all $E \in \mathcal{X}'$.

It is **compact**, if X' is a compact measure preserving system.

Theorem

Let X be a system. Exactly one of the followings is true:

1. X is weak mixing ("**pseudorandomness**");
2. X has a nontrivial compact factor ("**structure**").

Kronecker Factors

Notation. If $X = (X, \mathcal{X}, \mu, T)$, $\mathcal{X}_{AP} := \{A \in X \mid \mathbb{1}_A \in AP(X)\}$.

Claim. $\mathcal{X}_{AP}$ is a T-invariant sub-σ-algebra of $\mathcal{X}$. Therefore $(X, \mathcal{X}_{AP}, \mu, T)$ is a factor of X, called **Kronecker factor**.

Remark. The Kronecker is the maximal compact factor of X.

Proposition. Let $(X, \mathcal{X}, \mu, T)$ be a system, and $f \in L^2(X, \mathcal{X}, \mu)$.

1. $f \in AP(X)$ iff f is $\mathcal{X}_{AP}$-measurable: $AP(X) = L^2(X, \mathcal{X}_{AP}, \mu)$.
2. $f \in WM(X)$ iff $\mathbb{E}(f|\mathcal{X}_{AP}) = 0$ a. e.
3. **(♠)** We can write $f = f_{AP} + f_{WM}$, where $f_{AP} := \mathbb{E}(f|\mathcal{X}_{AP}) \in AP(X)$, and $f_{WM} := f - f_{AP} \in WM(X)$.