ASYMPTOTIC BEHAVIOR OF THE SMALLEST EIGENVALUE OF A SCHROEDINGER-TYPE OPERATOR WITH COULOMB POTENTIAL

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ABSTRACT. The aim of the rotation was to investigate the asymptotic behavior of the smallest eigenvalue of the Schroedinger-type operator $|p^2 - 1| - \lambda|x|^{-1}$ for small λ, using the Birman-Schwinger principle. In this draft we collect some preliminary results we got and some attempts we tried to develop during our discussions.

1. Setting of the problem

Let V be the Coulomb potential, i.e. $V(x) = -\frac{1}{|x|}$. Consider $K_E = |p^2 - 1| + E$ for small $E > 0$.

$$V^{\frac{1}{2}} \frac{1}{K_E} |V|^{\frac{1}{2}} = V^{\frac{1}{2}} \left[\frac{1}{K_E} - \frac{1}{p^2+1} \right] |V|^{\frac{1}{2}} + V^{\frac{1}{2}} \frac{1}{p^2+1} |V|^{\frac{1}{2}} \tag{1}$$

$$\begin{aligned} \int_{\mathbb{R}^3} e^{ip\cdot(x-y)} \left[\frac{1}{K_E} - \frac{1}{p^2+1} \right] \mathrm{d}^3 p = & \int_{\mathbb{R}^3} e^{i\frac{p}{|p|}\cdot(x-y)} \left[\frac{1}{K_E} - \frac{1}{p^2+1} \right] \mathrm{d}^3 p \\ & + \int_{\mathbb{R}^3} \left(e^{ip\cdot(x-y)} - e^{i\frac{p}{|p|}\cdot(x-y)} \right) \left[\frac{1}{K_E} - \frac{1}{p^2+1} \right] \mathrm{d}^3 p \end{aligned} \tag{2}$$

The first term of (2) can be rewritten as

$$\int_{\mathbb{R}^3} e^{i\frac{p}{|p|}\cdot(x-y)} \left[\frac{1}{K_E} - \frac{1}{p^2+1} \right] \mathrm{d}^3 p = m(E) \int_{\mathbb{S}^2} e^{i\omega\cdot(x-y)} \,\mathrm{d}\omega \tag{3}$$

where

$$m(E) := \int_0^{+\infty} \left[\frac{1}{K_E} - \frac{1}{p^2+1} \right] p^2 \,\mathrm{d}p \,. \tag{4}$$

Note that the integral on the unit sphere in (3) is nothing but the integral kernel of $\mathcal{F}^\dagger \mathcal{F}$, where $\mathcal{F} : L^1(\mathbb{R}^3) \to L^2(\mathbb{S}^2)$ is the Fourier transform defined as

$$(\mathcal{F}\phi)(p) := \frac{1}{2\pi^{\frac{3}{2}}} \int_{\mathbb{R}^3} \phi(x) e^{ip\cdot x} \,\mathrm{d}^3 x \,.$$

We denote by A_E the integral kernel of the second term of (2), thus we have

$$\frac{1}{K_E} - \frac{1}{p^2+1} = m(E)\mathcal{F}^\dagger \mathcal{F} + A_E \,.$$

Moreover, we define

$$M_E := \frac{1}{K_E} - m(E)\mathcal{F}^\dagger \mathcal{F} = A_E + \frac{1}{p^2+1} \,.$$

Our aim is to prove that $V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}^3)$ uniformly in E.
Indeed, in this case, $1 + \lambda V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}$ would be invertible for small $\lambda > 0$. Hence, we could write

$$\begin{aligned} 1 + \lambda V^{\frac{1}{2}} K_E^{-1} |V|^{\frac{1}{2}} = & 1 + \lambda V^{\frac{1}{2}} (m(E)\mathcal{F}^\dagger \mathcal{F} + M_E) |V|^{\frac{1}{2}} = \\ = & (1 + \lambda V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}) \left(1 + \frac{\lambda m(E)}{1 + \lambda V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}} V^{\frac{1}{2}} \mathcal{F}^\dagger \mathcal{F} |V|^{\frac{1}{2}} \right) . \end{aligned} \tag{5}$$

Note that $\lambda V^{\frac{1}{2}} K_E^{-1} |V|^{\frac{1}{2}}$ having eigenvalue -1 is equivalent to $\frac{\lambda m(E)}{1+\lambda V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}} V^{\frac{1}{2}} \mathcal{F}^\dagger \mathcal{F} |V|^{\frac{1}{2}}$ having eigenvalue -1. Moreover, $\frac{\lambda m(E)}{1+\lambda V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}} V^{\frac{1}{2}} \mathcal{F}^\dagger \mathcal{F} |V|^{\frac{1}{2}}$ is isospectral to the self-adjoint operator $\mathcal{F} |V|^{\frac{1}{2}} \frac{\lambda m(E)}{1+\lambda V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}} V^{\frac{1}{2}} \mathcal{F}^\dagger$ acting on $L^2(\mathbb{S}^2)$.

The Birman-Schwinger principle tell us that $\lambda V^{\frac{1}{2}} K_E^{-1} |V|^{\frac{1}{2}}$ having eigenvalue -1 is equivalent to $K_E + \lambda V$ having 0 as lowest eigenvalue, i.e. $|p^2 - 1| + \lambda V$ having $-E$ as lowest eigenvalue. As a consequence, the lowest eigenvalue $-E$ would satisfy

$$\lambda m(E) \inf \operatorname{spec} \mathcal{F} |V|^{\frac{1}{2}} \frac{\lambda m(E)}{1 + \lambda V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}} V^{\frac{1}{2}} \mathcal{F}^\dagger = -1 \,. \tag{6}$$

Up to first order in λ,

$$\lambda m(E) \inf \operatorname{spec} \mathcal{F} |V|^{\frac{1}{2}} [1 - \lambda V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}} + O(\lambda^2)] V^{\frac{1}{2}} \mathcal{F}^\dagger = -1, \tag{7}$$

where the error term $O(\lambda^2)$ is uniformly bounded in E. In particular, we would have

$$\lambda m(E) = \frac{-1}{\inf \operatorname{spec} \, [\mathcal{F} V \mathcal{F}^\dagger - \lambda \mathcal{F} V M_E V \mathcal{F}^\dagger + O(\lambda^2)]} \,. \tag{8}$$

Hence, if $V^{\frac{1}{2}} M_E |V|^{\frac{1}{2}}$ was uniformly bounded in E, it would follow that

$$\lim_{\lambda \to 0} \lambda m(E) = - \frac{1}{\inf \operatorname{spec} \, \mathcal{F} V \mathcal{F}^\dagger} \,. \tag{9}$$

2. Boundedness of $V^{\frac{1}{2}} (p^2+1)^{-1} |V|^{\frac{1}{2}}$

Let $B^* := \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |(x,y)| \le R^*\}$, $R^* > 0$, $B_{R^*} := \{x \in \mathbb{R}^3 : |x| \le R^*\}$ and $\chi^* := \chi_{B_{R^*}}$. Let $f_i \in L^2(\mathbb{R}^3)$, $i = 1, 2$.

$$\langle f_1 | V^{\frac{1}{2}} \frac{1}{p^2+1} |V|^{\frac{1}{2}} | f_2 \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_1(x) V^{\frac{1}{2}}(x) \frac{e^{-|x-y|}}{|x-y|} |V|^{\frac{1}{2}}(y) f_2(y) \, \mathrm{d}^3x \, \mathrm{d}^3y = I_a + I_b \tag{10}$$

Let $g(x) := \frac{e^{-|x|}}{|x|}$, $x \in \mathbb{R}^3$.

$$\begin{aligned} |I_a| &= \left| \int_{B^*} f_1(x) V^{\frac{1}{2}}(x) g(x-y) |V|^{\frac{1}{2}}(y) f_2(y) \, \mathrm{d}^3x \, \mathrm{d}^3y \right| \\ &\le \int_{\mathbb{R}^3 \times \mathbb{R}^3} |f_1(x) V^{\frac{1}{2}}|(x) \chi^*(x) g(x-y) |V|^{\frac{1}{2}}(y) |f_2|(y) \chi^*(y) \, \mathrm{d}^3x \, \mathrm{d}^3y \\ &\le K_{r,q,w;3} ||f_1 V^{\frac{1}{2}} \chi^*||_r ||f_2 |V|^{\frac{1}{2}} \chi^*||_q ||g||_{weak,w} \,, \end{aligned} \tag{11}$$

where we used the weak Young inequality for r, q, w satisfying $\frac{1}{r} + \frac{1}{q} + \frac{1}{w} = 2$.
Observe that $g \in L^3_{weak}(\mathbb{R}^3)$, so that we can choose $w = 3$. Indeed,

$$\begin{aligned} ||g||_{weak,w} &= \sup_A \frac{1}{|A|^{\frac{1}{w'}}} \int_A \frac{e^{-|x|}}{|x|} \, \mathrm{d}^3x = \sup_R cR^{-\frac{3}{w'}} \int_0^R \frac{e^{-r}}{r} r^2 \, \mathrm{d}r \\ &\le \sup_R cR^{-\frac{3}{w'}} \int_0^R \frac{1}{r} r^2 \, \mathrm{d}r = \sup_R cR^{-\frac{3}{w'}} R^2 \,, \end{aligned} \tag{12}$$

where w' is the conjugate exponent of w. Since we want $||g||_{weak,w}$ to be bounded, we choose $w' = \frac{3}{2}$, thus $w = 3$. As a consequence, for $w = 3$ and $r = q$, we get $r = \frac{6}{5}$.

$$||f_i V^{\frac{1}{2}} \chi^*||_{\frac{6}{5}} \le ||f_i||_2 ||V^{\frac{1}{2}} \chi^*||_3 \,, \; i = 1, 2. \tag{13}$$

Note that $V \in L^{\frac{3}{2}}(B_{R^*})$, hence $||V^{\frac{1}{2}}\chi^*||_3$ is bounded.
Let $\tilde{\chi} := \chi_{B_{\tilde{R}}}$, where $\tilde{R} < R^*$ such that $B_{\tilde{R}} \times B_{\tilde{R}} \subset B^*$.

$$\begin{aligned} |I_b| &= \left| \int_{\mathbb{R}^3\times\mathbb{R}^3 \setminus B^*} f_1(x)V^{\frac{1}{2}}(x)g(x-y)|V|^{\frac{1}{2}}(y)f_2(y)\,\mathrm{d}^3x\,\mathrm{d}^3y \right| \\ &\leq C_{r,q,p;3}||f_1V^{\frac{1}{2}}\tilde{\chi}||_r||f_2|V|^{\frac{1}{2}}\tilde{\chi}||_q||g||_p\,, \end{aligned} \tag{14}$$

where we used the Young inequality for r, q, p satisfying $\frac{1}{r} + \frac{1}{q} + \frac{1}{p} = 2$. For $p = 1$ and $r = q$, we get $r = 2$.

$$||f_iV^{\frac{1}{2}}\tilde{\chi}||_2 \leq ||f_i||_2||V^{\frac{1}{2}}\tilde{\chi}||_\infty\,,\ i = 1, 2. \tag{15}$$

Note that $V \in L^\infty(\mathbb{R}^3 \setminus B_{\tilde{R}})$, hence $||V^{\frac{1}{2}}\tilde{\chi}||_\infty$ is bounded.
Finally, since $V \in L^{\frac{3}{2}} + L^\infty$, we can conclude that $V^{\frac{1}{2}}\frac{1}{p^2+1}|V|^{\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}^3)$

3. Boundedness of $V^{\frac{1}{2}}A_E|V|^{\frac{1}{2}}$

Consider

$$\int_{\mathbb{R}^3} \left(e^{ip\cdot(x-y)} - e^{i\frac{p}{|p|}\cdot(x-y)}\right)\left[\frac{1}{K_E} - \frac{1}{p^2+1}\right]\mathrm{d}^3p\,.$$

By using spherical coordinates, we can rewrite it as

$$\begin{aligned} &2\pi\int_0^\infty \mathrm{d}p\,p^2\left[\frac{1}{|p^2-1|+E} - \frac{1}{p^2+1}\right]\int_0^\pi \mathrm{d}\theta\,\sin\theta(e^{ip\cos\theta|x-y|} - e^{i\cos\theta|x-y|}) = \\ &= 2\pi\int_0^\infty \mathrm{d}p\,p^2\left[\frac{1}{|p^2-1|+E} - \frac{1}{p^2+1}\right]\int_{-1}^1 \mathrm{d}r\,(e^{ipr|x-y|} - e^{ir|x-y|}) = \\ &= 2\pi\int_0^\infty \mathrm{d}p\,p^2\left[\frac{1}{|p^2-1|+E} - \frac{1}{p^2+1}\right]\left[\frac{\sin(p|x-y|)}{p|x-y|} - \frac{\sin(|x-y|)}{|x-y|}\right]. \end{aligned}$$

Let

$$\mathcal{A}_E(x) := \frac{2\pi}{|x|}\int_0^\infty \mathrm{d}p\,p^2\left[\frac{1}{|p^2-1|+E} - \frac{1}{p^2+1}\right]\left[\frac{\sin(p|x|)}{p} - \sin(|x|)\right]. \tag{16}$$

$\mathcal{A}_E$ is uniformly bounded in E. Indeed, since $\frac{\sin a}{a} - \frac{\sin b}{b} \leq c\frac{|a-b|}{|a+b|}$, then

$$|\mathcal{A}_E(x)| \leq c\int_0^\infty \mathrm{d}p\,p^2\left|\frac{1}{|p^2-1|+E} - \frac{1}{p^2+1}\right|\frac{|p-1|}{p+1}\,. \tag{17}$$

In particular,

$$|\mathcal{A}_0(x)| \leq c\int_0^\infty p^2\left|\frac{1}{(p+1)^2} - \frac{|p-1|}{p^2+1}\right|\mathrm{d}p\, < \infty\,, \tag{18}$$

because

$$p^2\left|\frac{1}{(p+1)^2} - \frac{|p-1|}{p^2+1}\right| \sim \frac{1}{p^2}$$

for large p.
However, the uniformly boundedness of $\mathcal{A}_E$ is not enough to have $V^{\frac{1}{2}}A_E|V|^{\frac{1}{2}}$ bounded on $L^2(\mathbb{R}^3)$ in the case of the Coulomb potential. This suggest us to study the behavior of $\mathcal{A}_E(x)$

for large $|x|$.

$$\begin{aligned}|\mathcal{A}_0(x)| \leq & \frac{2\pi}{|x|}\int_{1-\frac{1}{|x|}}^{1+\frac{1}{|x|}} p\left|\frac{1}{|p-1|(p+1)} - \frac{1}{p^2+1}\right| |\sin(p|x|) - p\sin(|x|) \pm \sin(|x|)|\, \mathrm{d}p \\ & + \frac{2\pi}{|x|}\int_0^{1-\frac{1}{|x|}} p^2\left|\frac{1}{|p^2-1|} - \frac{1}{p^2+1}\right|\left|1+\frac{1}{p}\right| \mathrm{d}p \\ & + \frac{2\pi}{|x|}\int_{1+\frac{1}{|x|}}^{+\infty} p^2\left|\frac{1}{|p^2-1|} - \frac{1}{p^2+1}\right|\left|1+\frac{1}{p}\right| \mathrm{d}p \\ \leq & \frac{2\pi}{|x|}\int_{1-\frac{1}{|x|}}^{1+\frac{1}{|x|}} p\frac{2}{(p+1)(p^2+1)} |\sin(|x|) + c|x||\, \mathrm{d}p \\ & + \frac{2\pi}{|x|}\int_0^{1-\frac{1}{|x|}} \frac{2p^2}{(p+1)(p^2+1)(1-p)}\left|1+\frac{1}{p}\right| \mathrm{d}p \\ & + \frac{2\pi}{|x|}\int_{1+\frac{1}{|x|}}^{+\infty} \frac{2p^2}{(p+1)(p^2+1)(p-1)}\left|1+\frac{1}{p}\right| \mathrm{d}p \\ \leq & \frac{2\pi}{|x|}[1+c|x|]\int_{1-\frac{1}{|x|}}^{1+\frac{1}{|x|}} \frac{2}{p^2+1}\, \mathrm{d}p \\ & + \frac{2\pi}{|x|}\int_0^{1-\frac{1}{|x|}} \frac{2p}{(p-1)(p^2+1)}\, \mathrm{d}p \\ & + \frac{2\pi}{|x|}\int_{1+\frac{1}{|x|}}^{+\infty} \frac{2p}{(p-1)(p^2+1)}\, \mathrm{d}p \\ \leq & \frac{2\pi}{|x|}[1+c|x|]\frac{4}{|x|} + \frac{2\pi}{|x|}c \end{aligned} \tag{19}$$

We thus deduce that $\mathcal{A}_E(x) \lesssim \frac{1}{|x|}$ for large $|x|$, but this decay is not enough for the Coulomb potential. Hence, we could try to estimate the behavior of $\mathcal{A}_E(x)$ for large $|x|$ in a sharper way without letting E going to zero. However, also in this case, long computations lead to

$$\begin{aligned}\mathcal{A}_E(x) \leq & -(2-E)\log\left(1 - \frac{1}{(\sqrt{1+E}+1)|x|}\right) + (2-E)\log\left(1 + \frac{1}{(\sqrt{1+E}+1)|x|}\right) \\ & -(2-E)\log\left(1 + \frac{1}{\sqrt{1+E}} - \frac{1}{(\sqrt{1+E}+1)|x|}\right) \\ & + \frac{c}{|x|}\frac{2-E}{\sqrt{1-E}}\left[\log\left(1-\sqrt{1-E}+\frac{1}{|x|}\right) - \log\left(1+\sqrt{1-E}+\frac{1}{|x|}\right)\right]\end{aligned} \tag{20}$$

for large $|x|$ and $E << 1$.

4. Another strategy

Consider directly

$$\int_{\mathbb{R}^3} e^{ip\cdot(x-y)}\left[\frac{1}{K_E} - \frac{1}{p^2+1}\right] \mathrm{d}^3p =: \mathcal{D}_E(x-y) \tag{21}$$

without splitting it into two terms as done in (2).

$$\langle f_1 \big| V^{\frac{1}{2}} \mathcal{D}_E |V|^{\frac{1}{2}} \big| f_2\rangle$$

$$= \int_{\mathbb{R}^3\times\mathbb{R}^3} f_1(x) V^{\frac{1}{2}}(x) \Big(\int_{\mathbb{R}^3} e^{ip\cdot(x-y)} \left[\frac{1}{|p^2-1|+E} - \frac{1}{p^2+1} \right] \mathrm{d}^3 p \Big) |V|^{\frac{1}{2}}(y) f_2(y)\ \mathrm{d}^3 x\, \mathrm{d}^3 y =$$

(22)

$$= \int_{\mathbb{R}^3} \left[\frac{1}{|p^2-1|+E} - \frac{1}{p^2+1} \right] \Big(\int_{\mathbb{R}^3} \frac{f_1(x)}{|x|^{\frac{1}{2}}} e^{ip\cdot x}\, \mathrm{d}^3 x \Big) \Big(\int_{\mathbb{R}^3} \frac{f_2(y)}{|y|^{\frac{1}{2}}} e^{-ip\cdot y}\, \mathrm{d}^3 y \Big)\, \mathrm{d}^3 p$$

From now on we will denote by $^\wedge$ the Fuorier transform operator and by $^\vee$ its inverse.

$$\int_{\mathbb{R}^3} \frac{f_1(x)}{|x|^{\frac{1}{2}}} e^{ip\cdot x}\, \mathrm{d}^3 x = \hat{f}_1(p) * \left[\frac{1}{|x|^{\frac{1}{2}}} \right]^\wedge = \hat{C}_{3,\frac{1}{2}} \hat{f}_1(p) * \frac{1}{|p|^{\frac{5}{2}}} =: g_1(p) \tag{23}$$

$$\int_{\mathbb{R}^3} \frac{f_2(y)}{|y|^{\frac{1}{2}}} e^{-ip\cdot y}\, \mathrm{d}^3 y = \check{f}_2(p) * \left[\frac{1}{|x|^{\frac{1}{2}}} \right]^\vee = \check{C}_{3,\frac{1}{2}} \check{f}_2(p) * \frac{1}{|p|^{\frac{5}{2}}} =: g_2(p) \tag{24}$$

We thus obtain

$$\langle f_1 \big| V^{\frac{1}{2}} \mathcal{D}_E |V|^{\frac{1}{2}} \big| f_2\rangle = C_{3,\frac{1}{2}} \int_{\mathbb{R}^3} \left[\frac{1}{|p^2-1|+E} - \frac{1}{p^2+1} \right] g_1(p) g_2(p)\, \mathrm{d}^3 p\,. \tag{25}$$

Notice that $|p|^{-\frac{5}{2}}$ is the Green's function for Δ^α when $\alpha = \frac{1}{4}$. Hence,

$$\Delta^{\frac{1}{4}} \Big(\hat{f}_i(p) * \frac{1}{|p|^{\frac{5}{2}}} \Big) = \hat{f}_i(p),\ i = 1, 2. \tag{26}$$

Furthermore, since $\hat{f}_i \in L^2(\mathbb{R}^3)$, then $g_i \in H^{\frac{1}{2}}_{loc}(\mathbb{R}^3) \hookrightarrow L^q_{loc}(\mathbb{R}^3)$, $q \leq 3$, $i = 1, 2$.
On the other hand, observe that

$$\left[\frac{1}{K_E} - \frac{1}{p^2+1} \right] \sim \frac{\chi_{B_R}(p)}{|p^2-1|+E} =: \tilde{g}_E(p)\,.$$

Hence,

$$\langle f_1 \big| V^{\frac{1}{2}} \mathcal{D}_E |V|^{\frac{1}{2}} \big| f_2\rangle \sim \int_{\mathbb{R}^3} \frac{\chi_{B_R}}{|p^2-1|+E} g_1(p) g_2(p)\, \mathrm{d}^3 p \leq ||\tilde{g}_E||_s ||g_1||_q ||g_2||_q\,, \tag{27}$$

where s,q satisfy $1 = \frac{2}{q} + \frac{1}{s}$, thus $s \geq 3$.
However,

$$||\tilde{g}_E||_s^s = \int_{B_R} \left[\frac{1}{|p^2-1|+E} \right]^s \mathrm{d}^3 p = c \int_0^R \frac{p^2}{(|p^2-1|+E)^s}\, \mathrm{d}p$$

$$\xrightarrow[E\to 0]{} c \int_0^R \frac{p^2}{(p-1)^s (p+1)^s}\, \mathrm{d}p = \infty \quad \text{for } s \geq 3\,. \tag{28}$$

Analogously, we could write

$$\langle f_1 \big| V^{\frac{1}{2}} A_E |V|^{\frac{1}{2}} \big| f_2\rangle \sim \int_{\mathbb{R}^3} \tilde{g}(|p|) \big[g_1(p) g_2(p) - g_1\big(\tfrac{p}{|p|}\big) g_2\big(\tfrac{p}{|p|}\big) \big]\, \mathrm{d}^3 p$$

$$\sim \int_0^\infty \tilde{g}(p) \Big(\int_{\mathbb{S}^2} \big[g_1(p\omega) g_2(p\omega) - g_1(\omega) g_2(\omega) \big]\, \mathrm{d}\omega \Big)\, \mathrm{d}p\,, \tag{29}$$

but also this does not seem to help. Indeed, we know that $g_i \in H^{\frac{1}{2}}_{loc}(\mathbb{R}^3)$, $i = 1, 2$. However, by using the Trace Theorem, we would only get the trace of g_i belonging to $L^2(\mathbb{S}^2)$.