Review on anyons

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1 Leinaas and Myrheim

The notion of fermions and bosons relies on the postulate of (anti)symmetry of the wave-function of a system of N identical quantum particles. If we were able to label in a coherent way these particles, then their identical nature would imply that:

$$|\psi(p(x_1 \ldots x_N))|^2 = |\psi(x_1 \ldots x_N)|^2 \; \forall p \in S_N. \tag{1}$$

Now, one could argue that labeling the particles corresponds to a non-observable operation, at least in quantum mechanics. Since it is impossible to distinguish them in the first place, how would it be possible to assign coherently some labels to each of them?

Starting from this question, one could look for a more a priori way to introduce indistinguishable particles. This is possible also in a classical frame, and indeed needed as we will see, and the scheme can later be extended to the quantum formalism.

1.1 Classical Mechanics

In this section, the classical description of a many body system is analyzed and indistinguishability is taken into account, basically redefining the configuration space of the particles.

1.1.1 Gibbs' Paradox

It was indeed clear to Gibbs, long before quantum mechanics came to change our lives, that in some situations it is necessary to take into account the iden-

tical nature of the particles. This gave rise to his discovery of the paradox that is named after him. He indeed argued that when calculating the entropy change in a process of mixing to identical gases (same volume, temperature etc.), in the thermodynamic limit, one should take into account that configurations that differ only for a permutation of the particles do not contribute to the entropy. If this is not considered, than the entropy of the mixed gas is bigger than the sum of the entropies of the two gases alone.

Example: let A be a system of N particles in a volume V and with some total energy E.

$$S(N, V, E) = k_B \log\left(\frac{w}{w_0}\right), \tag{2}$$

where w is the volume available in the phase space, while w_0 is an arbitrary scale factor. Now, w is the product of the volume in configuration space $u = V^N$ and in momentum space. The latter corresponds to the surface of a $3N$-dimensional sphere of radius $\sqrt{2mE}$:

$$v = A_{3N}(2mE)^{(3N-1)/2}. \tag{3}$$

The coefficients A_n are defined through a recurrency relation:

$$A_n = \frac{2\pi}{n-2} A_{n-2} \quad A_1 = 2 \quad A_2 = 2\pi. \tag{4}$$

One can see, through the previous definition, that for even n it takes exactly $n/2-1$ steps to get to A_2, whether for odd n it takes $(n-1)/2$ steps to get to A_1. In the first case, then, the numerator of eq.4 after all the iterations is exactly $(2\pi)^{n/2}$, while in the second it approximates the same value for large n. As this is the case we are interested in, the thermodynamic limit, we can just take this value, independent on the even or odd number of particles. Concerning the denominator, if we borrow the 2 from the numerator, is basically $(n/2 - 1)!$.

$$\begin{aligned} \log(A_n) &\approx \frac{n}{2}\log(\pi) + (\frac{n}{2} - 1)\log(2) - \log\left((\frac{n}{2} - 1)!\right) \\ &\approx \frac{n}{2}\log(\pi) + (\frac{n}{2} - 1)\log(2) - (\frac{n}{2} - 1)\log\left(\frac{n}{2} - 1\right) \\ &\approx \frac{n}{2}\left(\log(\pi) + 1 - \log\left(\frac{n}{2}\right)\right). \end{aligned} \tag{5}$$

From this result we reach the conclusion that the entropy of the system is:

$$S(N, V, E) = k_B \log\left(V^N A_{3N} (2mE)^{(3N-1)/2}\right) = k_B(N \log(V) + \log(A_{3N}) + \frac{3N-1}{2} \log(2mE))$$

$$\approx k_B(N \log(V) + \frac{3N}{2}(\log(\pi) + 1 - \log\left(\frac{3N}{2}\right) + \log(2mE))$$

$$= k_B(N \log(V) + N \log\left(\pi e \frac{4mE}{3N}\right)^{3/2}) = N k_B \log\left(CV(\frac{E}{N})^{3/2}\right).$$

$$(6)$$

Hence, we see that the entropy difference among the system of double size and the sum of the two systems is $2N k_B \log(2)$, which is wrong. On the other hand, if we take into account the Gibss' paradox, then we have a much smaller phase space $\frac{w}{N!}$. Using Stirling's approximation, we then get:

$$S'(N, E, V) \approx S(N, E, V) - N \log(N) \approx N k_B \log\left(C\frac{V}{N}(\frac{E}{N})^{3/2}\right). \quad (7)$$

Now the entropy difference in the thermodynamic limit is zero, as it should.

From this little example we see that also in classical mechanics indistinguishability of particles does play a fundamental role. Therefore it is legitimate to ask whether it is possible to include somehow classically the identical particle problem and quantize it later.

1.1.2 Configuration Space

In classical mechanics the dynamics of the system is described by equations of motion, a couple of differential equations which can be derived through Poisson's brackets from the Hamiltonian $H(\vec{x}, \vec{p})$. As discussed in the previous section, the phase space is the Cartesian product of the configuration and momentum space. Assuming N particles to be identical means that we cannot distinguish one from the other, and therefore all the elements of the configuration space which are different just for a permutation of indexes cannot be counted as distinct. Let X_N is the $N-$times Cartesian product of the one particle configuration space X with itself, then if we want to fulfill

the request of having identical particles our configuration space is reduced to X_N/S_N, where again S_N is the permutation group of N elements.

As S_N is discrete and indeed finite (it has $N!$ different elements) X_N/S_N is locally isomorphic to X_N, with the isomorphism to fail at each singularity point, i.e. two points in X_N which are identical under some element of S_N. Thanks to this local isomorphic nature, one can usually neglect the problem of identical particles in classical mechanics. For each trajectory due to time evolution in X_N/S_N there are $N!$ trajectories in X_N. If they do not cross, then there is no problem at all, as there will be no singularities and one can always map each of the trajectories in X_N to the one of X_N/S_N. If they do cross, one can still handle the problem asking that continuous derivative of the curves.

Usually, we will deal with particles living in some real space $\mathbb{R}^d$ with $d = 1, 2, 3$. We can always split the problem into two separate systems: the centre of mass and the relative motion. The centre of mass coordinate is:

$$\vec{X} = \frac{1}{N} \sum_i^N \vec{x_i}. \tag{8}$$

Therefore, by definition, it is a vector in $\mathbb{R}^d$ and it is invariant under the action of S_N. We can then write down the configuration space as the Cartesian product of $\mathbb{R}^d$ and of the space of relative motion, which has dimensions $d(N-1)$. The centre of mass space is, as already mentioned, invariant under permutations, thus, if we define $r(d, N)$ to be the space of relative coordinates of N partilces which has the same property, we can say that the configuration space of N identical particles is:

$$\mathbb{R}^{dN}/S_N = \mathbb{R}^d \times r(d, N). \tag{9}$$

Now we restrict to the case $N = 2$, in order to get some concrete result. Then $\vec{x_1}$ and $\vec{x_2}$ are the coordinates of the two particles in the space. The relative motion vector is defined as $\vec{x} = \vec{x_1} - \vec{x_2}$ and we can see that $p_{12}\vec{x} = -\vec{x}$, with $p_{12} \in S_2$. In order to define $r(d, 2)$, then, we need to obtain, for each dimensionality, the manifold for which sending $\vec{x_1} \leftrightarrow \vec{x_2}$ in real space leaves $\vec{x}$ invariant.

The $d = 1$ case is quite peculiar, as particles living on a line cannot exchange position without passing through one another. As the particles are identical, it does not make any difference which one comes first, and therefore

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the space $r(1, 2)$ can be identified by the relation $x_1 \leq x_2$, i.e. by the half-line bounded by the point $x_1 = x_2$.

Since now we have modified the configuration space, we should be careful to the definition of parallel transport of vectors. This defines how a vector, as the momentum for instance, is transported along a curve such as a trajectory of a system of particles. In $r(1, 2)$ a trajectory that encounters the boundary line $x_1 = x_2$ gets reflected and therefore parallel transport on that curve apparently is not possible. Nevertheless, this is just a matter of definition. If we define parallel transport of a vector in a way that allows for the component of the vector normal to the boundary to be inverted each time the trajectory *bounces* on the edge, then our situation perfectly fits the definition. Therefore, we find out that if a closed curve is reflected by the boundary line, we can get a vector different from the initial one.

For $d = 2$ we get that $r(2, 2)$ is given by the plane in which we identify $\vec{x}$ with $-\vec{x}$. A possible way to do so is to cut the plane along a line l from $0 \to \infty$ and then to wrap it on itself up to the point in which every point is in a one to one correspondence with its opposite. This is given by a cone, centered at the origin, with half-angle $\theta = 30$.
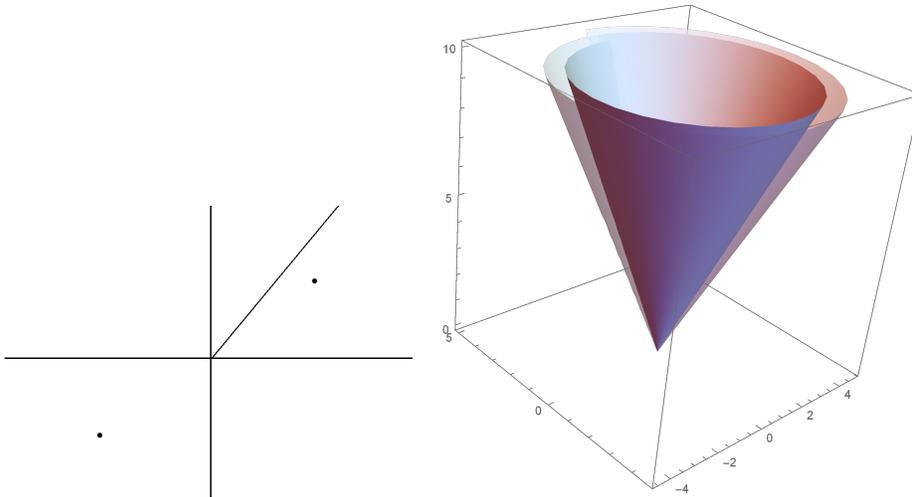
We can understand parallel transport mapping back the cone on the plane. If we draw a curve from $\vec{x}$ to $-\vec{x}$ and transport a vector $\vec{v}$ along this curve, we will see that, as $\vec{x}$ and $-\vec{x}$ coincide on the cone, the curve is indeed closed. The vector $\vec{v}$, though, when transported on the curve on the cone, will change direction, from $\vec{v}$ to $-\vec{v}$.

On the other hand, if we draw a curve closed on the plane (i.e. from $\vec{x}$ to $\vec{x}$) and then map it to the cone, $\vec{v}$ will be transported to itself. Therefore we found out that each vector transported over a closed loop in $r(2,2)$ is transformed in $(-1)^m \vec{v}$, where m is the number of times the curve encircles the vertex of the cone.

It can be seen that this property is indeed more general, and one can identify two different classes of curves in $r(d,2)$. The first one does not change a tangent vector during parallel transport, and coincides with a curve in the normal space $\mathbb{R}^{2d}$ that maps a point $(\vec{x_1}, \vec{x_2})$ with itself. On the other hand, the second kind of curves transports $\vec{v}$ into $-\vec{v}$. These ones correspond to curves in $\mathbb{R}^{2d}$ that connect $(\vec{x_1}, \vec{x_2})$ with $(\vec{x_2}, \vec{x_1})$.

1.2 Quantum Mechanics

The introduction of indistinguishable particles in a classical framework is convenient, as we can avoid to introduce a symmetry postulate. Hence, several advantages come from that, the configuration space is locally isometric to the non-identical case, at least if we are far enough from singularities. Therefore, we avoid the unphysical situation in which two very far apart particles are connected through the statistics, even if they cannot physically interact.

On the other hand, it makes quantization a bit harder, as now the classical configuration space is curved and has singularities. We need therefore to be careful in describing the formalism for parallel transport of tangent vectors, which affects the definition of the derivative on the space and therefore of momentum.

As a matter of fact, the most interesting dimensionality is $d = 2$, where weird statistics can arise. It is on the plane, in fact, that it is possible to have interpolating statistics among bosons and fermions, the so-called anyons. Therefore, we will stick to the two-dimensional case, from now on.

In order to correctly quantize the configuration space, we proceed as follows. First, for each point in the configuration space $\vec{x}$ we define a one dimensional Hilbert space h_x. Then, we represent each state of the system as a vector field $\vec{\Psi}$ on the configuration space that to each point $\vec{x}$ connects a vector $\vec{\Psi}(\vec{x})$.

Then, if we define a particular gauge, which corresponds to choosing a basis on a Hilbert space, we can define the function $\psi(\vec{x})$ that represents the

coordinate of $\vec{\Psi}$ on that particular basis in the point $\vec{x}$.

$$\vec{\Psi}(\vec{x}) = \psi(\vec{x})\vec{\chi}_x. \tag{10}$$

As, of course, the state $\vec{\Psi}$ must be independent on the choice of the basis, the function $\psi(\vec{x})$ has to be gauge dependent. Lets suppose we introduce a change of basis through a unitary transformation M: $\chi'_x = M_x \chi_x$.

$$\begin{aligned}
\Rightarrow \vec{\Psi}(\vec{x}) = \psi(\vec{x})\chi_x &= \psi'(\vec{x})\chi'_x \\
&= \psi'(\vec{x})M_x\chi_x \Rightarrow \psi'(\vec{x}) = (M_x)^{-1}\psi(\vec{x}) = e^{\imath\phi(\vec{x})}\psi(\vec{x}).
\end{aligned} \tag{11}$$

As mentioned above, we need to define a parallel transport of state vectors from h_x to $h_{x'}$. We call the operator which does this operation on a continuous line from $\vec{x}$ to $\vec{x}'$ $P(\vec{x}', \vec{x}) : h_x \to h_{x'}$. It is a linear and unitary operator. Usually, the result must depend on the particular pat connecting the points on the configuration space, but we assume that if $\vec{dx}$ is an infenitesimal displacement then $P(\vec{x} + \vec{dx}, \vec{x})$ is independent on the particular curve chosen and it is thus uniquely defined. We also assume that, for infinitesimal displacements, it is always possible to chose a gauge in such a way that the operator of parallel displacement has the following form.

$$P(\vec{x} + d\vec{x}, \vec{x})\chi_x = (1 + \imath dx^k b_k(\vec{x}))\chi_{x+dx}, \tag{12}$$

where Einstein notation in indexes summation is understood.

Now, in order to define a gauge-invariant differentiation operator in this gauge, we take the full derivative of the state vector, which is, by definition, gauge-invariant.

$$\begin{aligned}
D_k(\vec{\Psi}(\vec{x})) &= \chi_x D_k(\psi(\vec{x})) + \psi(\vec{x})D_k(\chi_x) \\
D_k(\chi_x) &= \lim_{dx^k \to 0} \frac{P(\vec{x} + dx^k, \vec{x})\chi_x - \chi_x}{dx^k} = \lim_{dx^k \to 0} \frac{(1 + \imath dx^k b_k(\vec{x}))\chi_{x+dx} - \chi_x}{dx^k} = \frac{\partial}{\partial x^k}\chi_x + \imath b_k(\vec{x})\chi_x \\
\Rightarrow D_k(\vec{\Psi}(\vec{x})) &= \chi_x D_k(\psi(\vec{x})) + \psi(\vec{x})\frac{\partial}{\partial x^k}\chi_x + \psi(\vec{x})\imath b_k(\vec{x})\chi_x.
\end{aligned} \tag{13}$$

In order for the derivative to be gauge-invariant, then, one needs the differentiation operator for the functions $\psi(\vec{x})$ to be:

$$D_k\psi(\vec{x}) = \frac{\partial}{\partial x^k}\psi(\vec{x}) - \imath b_k(\vec{x})\psi(\vec{x}). \tag{14}$$

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The functions $b_k(\vec{x})$ must be real, so that the operator $P(\vec{x}+d\vec{x})$ is unitary. We can also define a quantity measuring the commutativity of the different components of D:

$$f_{kl} = \imath[D_k, D_l] = \imath\big[(\partial_k - \imath b_k)(\partial_l - \imath b_l) - (\partial_l - \imath b_l)(\partial_k - \imath b_k)\big] = \frac{\partial b_l}{\partial x^k} - \frac{\partial b_k}{\partial x^l}. \quad (15)$$

We cannot help noticing the similarity with the gauge-invariance introduced in the magnetic vector potential formalism. Analogously, here f_{kl} plays the role of the force field, and b_k of the vector potential. As in this case we are not dealing with any force field, we require $f_{kl} = 0$ except at the singularities. Consequently, the state vector $\vec{\Psi}$ will be changed only by closed loops that encircle the singularity. Similarly to the case of normal vectors described in the previous section, the state vector here will change according to P_x^m if it rounds m times the singularity. Here P_x is a linear unitary operator acting on h_x. Since h_x is one-dimensional, P_x can only be a phase factor:

$$P_x = e^{\imath\xi(\vec{x})}. \quad (16)$$

And ξ is real. Being just a phase factor it commutes with the displacement operator and then

$$P_{x'} = P(\vec{x}', \vec{x})P_x P(\vec{x}', \vec{x})^{-1} = P_x \quad (17)$$

hence ξ is actually position independent.

We then conclude that if we have two identical particles, the state of the system after encircling m times the singularity of the configuration space, which means basically exchanging m times the two particles in the real space, will be acquire a phase $e^{\imath m\xi}$.

We can then already see that from the only request of having a configuration space that takes into account the indistinguishability of the particles we have the limits for bosons and fermions ($\xi = 0$ and $\xi = \pi$ respectively) without imposing any symmetry postulate.

Now we can define two different approaches depending on where do we introduce the dynamical effect of the particular topology of the configuration space. Either we introduce the field b_k in the gauge-invariant differentiation operator and therefore in the Hamiltonian, or we find a particular gauge in which $b_k = 0$, but in this latter case, we shall introduce a multivalued wave function, including the factor ξ.

1.2.1 Harmonic Oscillator

We want to study the problem of two identical particles, in two dimensions, that interact through a harmonic potential and satisfy the previous statistics for an arbitrary ξ. We shall call these kind of particles anyons.

As the harmonic potential depends only on the relative distance of the two particles, we can separate the centre of mass from the relative motion.

$$\vec{R} = \frac{\vec{x}_1 + \vec{x}_2}{2} \; ; \; \vec{r} = \vec{x}_1 - \vec{x}_2$$
$$\vec{P} = \vec{p}_1 + \vec{p}_2 \; ; \; \vec{p} = \frac{\vec{p}_1 - \vec{p}_2}{2}. \tag{18}$$

Therefore, introducing the total mass $M = 2m$ and the reduced mass $\mu = \frac{m}{2}$, we can see how the Hamiltonian splits into two independent terms:

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu}. \tag{19}$$

In the plane, using polar coordinates, the Hamiltonian describing the free relative motion, has the following form:

$$H = -\frac{\hbar^2}{2\mu} \Big[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \Big]. \tag{20}$$

As we learned from the previous section, we can have the simple normal Hamiltonian, but then we should take into account the multi-valued nature of the wavefunction. In this first case, eq.20 holds, but the wavefunctions must satisfy

$$\psi(r, \phi + 2n\pi) = e^{\imath n \xi} \psi(r, \phi). \tag{21}$$

Again we see that the parameter ξ gives the usual bosonic and fermionic limit, but there is no reason to restrict to these two cases.

On the other hand, we can build a single valued wavefunction through a unitary transformation, and from that very transformation define a new Hamiltonian, which will now contain the effect of the statistics in a part of the differentiation operator. Starting from the wavefunctions of eq.21, we define

$$\psi'(r, \phi) = e^{-\imath \xi \frac{\phi}{2\pi}} \psi(r, \phi). \tag{22}$$

It is easy to verify that this second function is indeed single-valued: $\psi'(r, \phi + 2\pi) = \psi'(r, \phi)$. The Hamiltonian hence transforms as:

$$H' = e^{-i\xi \frac{\phi}{2\pi}} H e^{i\xi \frac{\phi}{2\pi}}, \qquad (23)$$

the radial derivatives and the transformation $U_\phi = e^{i\xi \frac{\phi}{2\pi}}$ commute, thus just the angular derivative will be affected by the transformation. It means that the field b_k is indeed a b_ϕ field.

$$\frac{\partial^2}{\partial\phi^2} U_\phi = \frac{\partial}{\partial\phi}(\frac{\partial}{\partial\phi} U_\phi) = \frac{\partial}{\partial\phi}(\frac{\partial U_\phi}{\partial\phi} + U_\phi \frac{\partial}{\partial\phi}) = \frac{\partial^2 U_\phi}{\partial\phi^2} + 2\frac{\partial U_\phi}{\partial\phi}\frac{\partial}{\partial\phi} + U_\phi \frac{\partial^2}{\partial\phi^2} \quad (24)$$

The free particle Hamiltonian for anyons therefore is:

$$H' = -\frac{\hbar^2}{2\mu}\Big[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\big(\frac{\partial}{\partial\phi} + i\frac{\xi}{2\pi}\big)^2\Big]. \qquad (25)$$

If we now ask the two particles to interact via a harmonic potential $V(r) = \mu\omega^2 r^2$ the Hamiltonian of the system becomes $H = H' + V$. As V depends only on the modulus of the radius r and not on the angle, the total Hamiltonian H commutes with the projection of the momentum on ϕ, $[H, p_\phi] = 0$, and we can write the Eigenfunction as $\psi(r, \phi) = e^{il\phi}R(r)$. Then, the angular dependence of ψ is trivial and we can write the differential equation for $R(r)$.

$$\Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\big(l + \frac{\xi}{2\pi}\big)^2 - \frac{\mu^2\omega^2 r^2}{\hbar^2} + \frac{2\mu E}{\hbar^2}\Big]R(r) = 0. \qquad (26)$$

Apart from the presence of ξ, this is the equation for the harmonic oscillator in $2d$ in polar coordinates. We therefore proceed as in that case in order to find the Eigenvalues. First of all, we notice that $r \geq 0$ and therefore we can exchange r with the following dimensionless quantity without loss of generality: $x = \frac{\mu\omega}{\hbar}r^2$.

$$\begin{aligned}\frac{\partial^2}{\partial r^2} &= 2\frac{\mu\omega}{\hbar}\frac{\partial}{\partial x} + 4\frac{\mu\omega}{\hbar}x\frac{\partial^2}{\partial x^2} \\ \frac{1}{r}\frac{\partial}{\partial r} &= 2\frac{\mu\omega}{\hbar}\frac{\partial}{\partial x}\end{aligned}. \qquad (27)$$

Pluggin eqs.27 into the Hamiltonian, we get:

$$(H - E)R(x) = \Big[4\frac{\mu\omega}{\hbar}x\frac{\partial^2}{\partial x^2} + 4\frac{\mu\omega}{\hbar}\frac{\partial}{\partial x} - \frac{\mu\omega}{\hbar}\frac{\big(l + \frac{\xi}{2\pi}\big)^2}{x} - \frac{\mu\omega}{\hbar}x + \frac{\mu\omega}{\hbar}\epsilon\Big]R(x) = 0, \qquad (28)$$

where $\epsilon := \frac{2E}{\hbar\omega}$.

$$\Downarrow$$

$$(H-E)R(x) = \left[\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} - \frac{\left(l + \frac{\xi}{2\pi}\right)^2}{4x^2} - \frac{1}{4} + \frac{\epsilon}{4x}\right]R(x) = 0. \tag{29}$$

We now assume the following ansatz on the form of $R(x)$:

$$R(x) = x^{\frac{|l+\xi/2\pi|}{2}} e^{-x/2} \rho(x), \tag{30}$$

which gives

$$\frac{\partial R(x)}{\partial x} = x^{\frac{|l+\xi/2\pi|}{2}} e^{-x/2} \left[\left(\frac{|l+\xi/2\pi|}{2x} - \frac{1}{2}\right)\rho(x) + \rho'(x)\right] \tag{31}$$

and

$$\frac{\partial^2 R(x)}{\partial x^2} = x^{\frac{|l+\xi/2\pi|}{2}} e^{-x/2} \left[\left(\left(\frac{|l+\xi/2\pi|}{2x} - \frac{1}{2}\right)^2 - \frac{|l+\xi/2\pi|}{2x^2}\right)\rho(x) + \left(\frac{|l+\xi/2\pi|}{x} - 1\right)\rho'(x) + \rho''(x)\right]. \tag{32}$$

Therefore, plugging eq.32 and eq.31 into eq.29, we get:

$$\frac{1}{2}\left[\frac{\epsilon}{2} - |l + \frac{\xi}{2\pi}| - 1\right]\rho(x) + \left[|l + \frac{\xi}{2\pi}| - x + 1\right]\rho'(x) + x\rho''(x) = 0. \tag{33}$$

This last equation is known in the literature and corresponds to the generalized Laguerre equation. In order for it to have well behaved solutions, the following condition must be fulfilled.

$$\frac{1}{2}\left[\frac{\epsilon}{2} - |l + \frac{\xi}{2\pi}| - 1\right] = n \ ; \ \ n \in \mathbb{N} \ n \geq 0 \tag{34}$$

Then we see that the value of ξ, i.e. the statistics of the system, influences the eigenvalues shifting them:

$$E_{n,l}(\xi) = 2\hbar\omega\left(n + |l + \frac{\xi}{2\pi}| + \frac{1}{2}\right). \tag{35}$$

2 Anyons and quantum groups

It is well known that starting from the commutation and anti-commutation relations of Bosons and Fermions it is possible to realize the $SU(2)$ algebra, following Schwinger construction. This consists of constructing ladder spin

operators starting from annihilation and creation operators. Thanks to the commutation relations, then, the $SU(2)$ algebra is easily seen to be fulfilled.

A similar approach can be followed in order to construct a $SU(2)_q$ quantum group starting from anyonic oscillators and from their deformed commutation relations, which depend on their statistics ν. Therefore, in the following, we will define the anyonic creation and annihilation operators through a Jordan-Wigner transformation of fermionic operators. Once defined in such a way, the deformed commutation relations for anyonic oscillators follow quite easily.

2.1 Anyonic oscillators in $\mathbb{R}^2$

In order to perform a Jordan-Wigner transformation in a two-dimensional space, we need to define an *angle function*. In order to make the notation clear, we will briefly introduce properties and derivation of the angle function on $\mathbb{R}^2$, with x_1 and x_2 defining the two components of a vector $\vec{x}$ in $\mathbb{R}^2$.

The angle function on $\mathbb{R}^2$ can be formally defined through the Green function af the Laplacian:

$$G(\vec{x}, \vec{y}) = log(|\vec{x} - \vec{y}|). \tag{36}$$

$$\Delta G(\vec{x}, \vec{y}) = 2\pi\delta(\vec{x} - \vec{y}). \tag{37}$$

We can then introduce a vector field $\vec{f}(\vec{x}, \vec{y}) = (f^1(\vec{x}, \vec{y}), f^2(\vec{x}, \vec{y}))$, with components related to the Green function through the following equation:

$$f^i = -\epsilon^{ij}\frac{\partial}{\partial x^j}G(\vec{x}, \vec{y}), \tag{38}$$

so that $f^1 = -\frac{\partial G(\vec{x},\vec{y})}{\partial x^2}$ and $f^2 = \frac{\partial G(\vec{x},\vec{y})}{\partial x^1}$, where we used the completely antisymmetric tensor ϵ^{ij}. It is then possible to define the angle function $\Theta(\vec{x}, \vec{y})$ as:

$$\frac{\partial}{\partial x^i}\Theta(\vec{x}, \vec{y}) = f_i(\vec{x}, \vec{y}). \tag{39}$$

We can easily check that Θ satisfies the following relation:

$$\epsilon^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Theta(\vec{x}, \vec{y}) = \Delta G(\vec{x}, \vec{y}) = 2\pi\delta(\vec{x} - \vec{y}). \tag{40}$$

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Equation 40 has a solution which coincides with the usual definition of the angle between $\vec{x}$ and $\vec{y}$:

$$\Theta(\vec{x}, \vec{y}) = \arctan\left(\frac{y_2 - x_2}{y_1 - x_1}\right).$$ (41)

As this function is multivalued, it is necessary to introduce a branch cut, for instance the line $(x_1, -\infty)$, so that the values of Θ are restricted in the interval $[-\pi, \pi)$. It is easy to verify, especially graphically as shown in fig.1, that the following relation holds:

$$\Theta(\vec{x}, \vec{y}) - \Theta(\vec{y}, \vec{x}) = \begin{cases} \pi\, sign(y_2 - x_2), & \text{if } x_2 \neq y_2 \\ \pi\, sign(x_1 - y_1), & \text{if } x_2 = y_2 \end{cases}$$ (42)
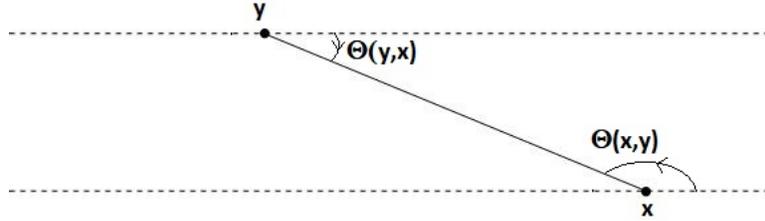

Figure 1: Angle function between $\vec{x}$ and $\vec{y}$. As the values of Θ are restricted in the interval $[-\pi, \pi)$ the two angles have opposite sign.

2.1.1 Jordan-Wigner transformation: anyonic operators on $\mathbb{R}^2$

We can finally define anyonic creation and annihilation operators on the plane through the Jordan-Wigner transformation, making use of the angle function previously defined. We must remark that the choice of the angle function is somewhat arbitrary, due to the various possible choices of the branch cut. We will consider just spinless anyons, in order to make the notation easier,

but we warn that the construction of the quantum group $SU(2)_q$ requires the presence of two different spins. We refer the interested reader to the original paper [2] for further details.

The way we will define anyonic operators is the following: starting from fermionic operators, we will transform them in order to make them non-local and to take into account the fact that different braidings lead to different final states. We will do so by introducing an operator $K(\vec{x})$ which basically *counts* the total number of particles and sticks to each of them a phase, given by the angle function $\Theta(\vec{x}, \vec{y})$.

$$K(\vec{x}) = \exp\left(\imath\nu \int_{\mathbb{R}^2} \Theta(\vec{x}, \vec{y}) c^\dagger(\vec{y}) c(\vec{y}) dy \right) \tag{43}$$

$$K^\dagger(\vec{x}) = \exp\left(-\imath\nu \int_{\mathbb{R}^2} \Theta(\vec{x}, \vec{y}) (c^\dagger(\vec{y}) c(\vec{y}))^\dagger dy \right) = (K(\vec{x}))^{-1} \tag{44}$$

where ν is a real parameter.

Given these two operators, we can then define creation and annihilation operators for anyons:

$$a(\vec{x}) = K(\vec{x}) c(\vec{x}) \tag{45}$$

$$a^\dagger(\vec{x}) = c^\dagger(\vec{x}) K^\dagger(\vec{x}) \tag{46}$$

with c and $c^\dagger$ are spinless fermionic operators.

In order to prove deformed commutation relation of these operators, we need before some useful equations:

$$
\begin{aligned}
K(\vec{x}) c(\vec{y}) &= \exp\left(\imath\nu \int_{\mathbb{R}^2} \Theta(\vec{x}, \vec{y}') c^\dagger(\vec{y}') c(\vec{y}') dy' \right) c(\vec{y}) \\
&= \left(1 + \imath\nu \int_{\mathbb{R}^2} \Theta(\vec{x}, \vec{y}') c^\dagger(\vec{y}') c(\vec{y}') dy' + O(\nu^2)\right) c(\vec{y}) \\
&= c(\vec{y}) - \imath\nu \int_{\mathbb{R}^2} \Theta(\vec{x}, \vec{y}') c^\dagger(\vec{y}') c(\vec{y}) c(\vec{y}') dy' \\
&= c(\vec{y}) - \imath\nu \int_{\mathbb{R}^2} \Theta(\vec{x}, \vec{y}') (\delta(\vec{y} - \vec{y}') - c(\vec{y}) c^\dagger(\vec{y}')) c(\vec{y}') dy' \\
&= c(\vec{y}) - c(\vec{y}) \imath\nu \Theta(\vec{x}, \vec{y}) + c(\vec{y}) \imath\nu \int_{\mathbb{R}^2} \Theta(\vec{x}, \vec{y}') c^\dagger(\vec{y}') c(\vec{y}') dy' \\
&= \exp(-\imath\nu \Theta(\vec{x}, \vec{y})) c(\vec{y}) K(\vec{x}),
\end{aligned}
\tag{47}
$$

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where we used the Taylor expansion of the exponential, assuming $\nu \ll 1$. This works also for the generic ν case, in which we can think of $K(\vec{x})$ as the product of infinite operators with infinitesimal ν, each satisfying the previous relation.

In a similar way, these relations follow:

$$
\begin{aligned}
K(\vec{x})c^\dagger(\vec{y}) &= \exp(\imath\nu\Theta(\vec{x},\vec{y}))c^\dagger(\vec{x})K(\vec{x}) \\
K(\vec{x})K(\vec{y}) &= K(\vec{y})K(\vec{x}).
\end{aligned}
\tag{48}
$$

From these equations we can see that the following deformed commutation relation holds, using the relation eq.42 for Θ:

$$
\begin{aligned}
a(\vec{x})a(\vec{y}) &= K(\vec{x})c(\vec{x})K(\vec{y})c(\vec{y}) \\
&= \exp(\imath\nu\Theta(\vec{y},\vec{x}))K(\vec{x})K(\vec{y})c(\vec{x})c(\vec{y}) \\
&= -\exp(\imath\nu\Theta(\vec{y},\vec{x}))K(\vec{y})K(\vec{x})c(\vec{y})c(\vec{x}) \\
&= -\exp(-\imath\nu(\Theta(\vec{x},\vec{y})-\Theta(\vec{y},\vec{x})))K(\vec{y})c(\vec{y})K(\vec{x})c(\vec{x}) \\
&= -\exp(-\imath\nu\pi)a(\vec{y})a(\vec{x})
\end{aligned}
\tag{49}
$$

Therefore we can write:

$$
a(\vec{x})a(\vec{y}) + q^{-1}a(\vec{y})a(\vec{x}) = 0,
$$

with $q = \exp(\imath\nu\pi)$ defining the statistics of the particles. In particular, we see that when $\nu = 0$ and $\nu = 1$ fermionic anticommutation relations and bosonic commutation relations are restored, respectively.

By hermitian conjugation we get also (as $q^* = q^{-1}$):

$$
a^\dagger(\vec{x})a^\dagger(\vec{y}) + q^{-1}a^\dagger(\vec{y})a^\dagger(\vec{x}) = 0.
\tag{50}
$$

With similar arguments to the ones used in eq.49, one can derive also

$$
a(\vec{x})a^\dagger(\vec{y}) + qa^\dagger(\vec{y})a(\vec{x}) = 0.
\tag{51}
$$

We want to stress here that these commutation relations reflect the statistics of the braid group, as exchanging particles in $\vec{x}$ and $\vec{y}$ gives opposite phases depending on if we do so clockwise or counterclockwise (i.e. if $x > y$ or $y < x$, given the cut that defines the angle function).

Commutation relations among creation and annihilation operators at the same point $\vec{x}$ deserve special attention, as here the angle function is not well defined. Nevertheless, as we shall see in the next lines, when we evaluate $a(\vec{x})a^\dagger(\vec{x})$ the angle function cancel out, yielding a well defined result.

$$
\begin{aligned}
a(\vec{x})a^\dagger(\vec{x}) &= K(\vec{x})c(\vec{x})c^\dagger(\vec{x})K^\dagger(\vec{x}) \\
&= K(\vec{x})(1 - c^\dagger(\vec{x})c(\vec{x}))K^\dagger(\vec{x}) \\
&= 1 - \exp(\imath\nu\Theta(\vec{x},\vec{x}))c^\dagger(\vec{x})K(\vec{x})c(\vec{x})K^\dagger(\vec{x}) \\
&= 1 - \exp(\imath\nu\Theta(\vec{x},\vec{x}))c^\dagger(\vec{x})K(\vec{x})\exp(-\imath\nu\Theta(\vec{x},\vec{x}))K^\dagger(\vec{x})c(\vec{x}) \\
&= 1 - c^\dagger(\vec{x})K^\dagger(\vec{x})K(\vec{x})c(\vec{x}) = 1 - a^\dagger(\vec{x})a(\vec{x})
\end{aligned}
\tag{52}
$$

from which we observe that anyons with statistics parameter ν obey standard anticommutation relations at the same point $\vec{x}$, as normal fermions.

We therefore see that starting from the Jordan-Wigner transformation defined in eq.45 one can obtain commutation relations deformed via a factor $q \in [-1, 1]$, that interpolates between bosons and fermions. As mentioned above, it is possible to build a quantum group out of these anyonic oscillators, in a way similar to the one used to get $SU(2)$ from bosonic and fermionic operators. We address the interest reader to reference [2] for further analysis.

References

[1] Leinaas J.M. and Myrheim J., *On the Theory of Identical Particles*; Il Nuovo Cimento **37 B** (1977)

[2] Lerda A. and Sciuto S., *Anyons and Quantum Groups*; Nuclear Physics B **B401** (1993)