Rotation Project in Laszlo Erdős Group

MULTI-RESOLVENT LOCAL LAWS FOR DEFORMED WIGNER MATRICES

OLEKSII KOLUPAIEV†

1. Introduction

The rotation project was a part of a work on the equipartition principle for Wigner matrices done by the author in collaboration with Giorgio Cipolloni, Laszlo Erdős and Joscha Henheik. The results of this bigger project are presented in [2]. Most of the content of the current report can be found in [2, Section 4]. However, here we discuss the proof of Lemma 3.2 in more details and give an alternative description of M in Appendix A.

It is well-known that resolvents of large random matrices are typically well approximated by their deterministic counterparts both in averaged and in isotropic sense. Such results are called local laws. Moreover, one can look at products of several resolvents and ask if they also tend to be deterministic when the size of a random matrix goes to infinity. It turns out that the answer is positive for Wigner matrices [4] and for Hermitization of matrices with independent identically distributed entries [3]. The main goal of the rotation project was to prove the similar result for deformed Wigner matrices by using the Ψ-approach from [4] and [3].

2. Set-up and main results

2.1. Model.

Definition 2.1. *Let W be a real symmetric or complex Hermitian Wigner matrix, i.e. a matrix with independent entries up to the symmetry constraint satisfying $w_{ab} = N^{-1/2}\mathcal{X}_{od}$, for $a > b$, and $w_{aa} = N^{-1/2}\mathcal{X}_d$. Let D be a bounded deterministic matrix of the same size with the corresponding symmetry type. Then the matrix $H := W + D$ is called a deformed Wigner matrix.*

We will need to make the following technical assumption about W:

Assumption 2.2. *We assume that χ_{d} is a real centered random variable, that χ_{od} is a real or complex random variable such that $\mathbf{E}\chi_{\mathrm{od}} = 0$ and $\mathbf{E}|\chi_{\mathrm{od}}|^2 = 1$; additionally in the complex case we also assume that $\mathbf{E}\chi_{\mathrm{od}}^2 = 0$. Furthermore, we assume that all the moments of χ_{od} and χ_{d} exist, i.e. for any $p \in \mathbf{N}$ there exists a constant $C_p > 0$ such that*

$$\mathbf{E}|\chi_{\mathrm{od}}|^p + \mathbf{E}|\chi_{\mathrm{d}}|^p \le C_p. \tag{2.1}$$

In the sequel deformation D will be fixed but N-dependent and we will omit the dependence on D in notations.

2.2. Notations and conventions.

For positive quantities f, g we write $f \lesssim g$ and $f \sim g$ if $f \le Cg$ or $cg \le f \le Cg$, respectively, for some constants $c, C > 0$ which depend only on the constants appearing in the moment condition, see (2.1).

We denote vectors by bold-faced lower case Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, for some $N \in \mathbf{N}$. Vector and matrix norms, $\|\boldsymbol{x}\|$ and $\|A\|$, indicate the usual Euclidean norm and the corresponding induced matrix norm. For any $N \times N$ matrix A we use the notation $\langle A \rangle := N^{-1}\mathrm{Tr}A$ to denote the normalized trace of A. Moreover, for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ and matrices $A \in \mathbf{C}^{N\times N}$ we define the scalar product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^N \overline{x}_i y_i\,.$$

†IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria
E-mail address: oleksii.kolupaiev@ist.ac.at.
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Finally, we will use the concept of "with very high probability" meaning that for any fixed $D > 0$ the probability of an N-dependent event is bigger than $1 - N^{-D}$ for $N \geq N_0(D)$. We introduce the notion of *stochastic domination* (see e.g. [5]): given two families of non-negative random variables

$$X = \left(X^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)}\right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)}\right)$$

indexed by N (and possibly some parameter u in some parameter space $U^{(N)}$), we say that X is stochastically dominated by Y, if for all $\xi, D > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^{\xi} Y^{(N)}(u)\right] \leq N^{-D} \tag{2.2}$$

for large enough $N \geq N_0(\xi, D)$. In this case we use the notation $X \prec Y$ or $X = \mathcal{O}_{\prec}(|Y|)$. We also use the convention that $\xi > 0$ denotes an arbitrary small constant which is independent of N.

2.3. **Main results.** Let D be a self-adjoint $N \times N$ matrix, $z \in \mathbb{C}\backslash\mathbb{R}$. The *matrix Dyson equation* (MDE) for deformed Wigner matrices is defined as follows:

$$-\frac{1}{M(z)} = z - D + \langle M(z) \rangle. \tag{2.3}$$

It is known that MDE has a solution and for z in the upper half-plane under assumption $\Im M(z) > 0$ it is unique. In the sequel we will denote this solution by $M(z) = M^D(z)$. It turns out that $M(z)$ approximates the resolvent $G(z) := (H - z)^{-1}$ in the sense of the following local laws:

$$|\langle (G(z) - M(z))A \rangle| \prec \frac{1}{N|\Im z|}, \tag{2.4}$$

$$|\langle \boldsymbol{x}, (G(z) - M(z))\boldsymbol{y} \rangle| \prec \frac{1}{\sqrt{N|\Im z|}} \tag{2.5}$$

in the regime $|\Im z| \gg \frac{1}{N}$ for any bounded matrix A and vectors $\boldsymbol{x}, \boldsymbol{y}$ (see e.g. [1]). Next we define the deterministic approximation of the product chains of the form

$$G(z_1)A_1G(z_2)A_2 \cdots A_{k-1}G(z_k), \tag{2.6}$$

where $z_1, z_2, \ldots, z_k$ are spectral parameters and matrices $A_1, A_2, \ldots, A_{k-1}$ are deterministic.

Definition 2.3. *The $N \times N$ matrix $M(z_1, A_1, z_2, A_2, \ldots, A_{k-1}, z_k)$ which we call the deterministic approximation of the product* (2.6) *is the solution of the following recurrence relation:*

$$\begin{aligned} M(z_1, A_1, \ldots, A_{k-1}, z_k) &= M(z_1)A_1M(z_2, A_2, \ldots, A_{k-1}, z_k) + \\ &+ \sum_{j=2}^{k-1} \langle M(z_1, A_1, \ldots, A_{j-1}, z_j) \rangle M(z_1) M(z_j, A_j, \ldots, A_{k-1}, z_k) + \\ &+ \langle M(z_1, A_1, \ldots, A_{k-1}, z_k) \rangle M(z_1) M(z_2). \end{aligned} \tag{2.7}$$

In Appendix A we prove that (2.7) defines M uniquely and give a more explicit description of the deterministic approximation. In particular, this description implies that M satisfies another recurrence relation which will be used further in calculations:

$$\begin{aligned} M(z_1, A_1, \ldots, A_{k-1}, z_k) &= M(z_1, A_1M(z_2)A_2, z_3, A_3, \ldots, A_{k-1}, z_k) + \\ &+ \langle M(z_1)A_1M(z_2) \rangle M(z_1, I, z_2, A_2, z_3, A_3, \ldots, A_{k-1}, z_k) + \\ &+ \langle M(z_2, A_2, z_3) \rangle M(z_1, A_1M(z_2), z_3, A_3, \ldots, A_{k-1}, z_k) + \cdots + \\ &+ \langle M(z_2, A_2, z_3, \ldots, A_{k-1}, z_k) \rangle M(z_1, A_1M(z_2), z_k). \end{aligned} \tag{2.8}$$

Our goal is to estimate the difference

$$G(z_1)A_1G(z_2)A_2 \cdots A_{k-1}G(z_k) - M(z_1, A_1, z_2, A_2, \ldots, A_{k-1}, z_k)$$

both in averaged and in isotropic sense. Note that in Wigner case this fluctuation becomes much smaller if some of the matrices A_j are traceless. A similar phenomenon is observed also for deformed Wigner matrices, but the condition on A_j which replaces tracelessness depends on a place of A_j in the product chain. More precisely, it depends on z_j and z_{j+1}. This leads us to the definition of *regular observable* (we sometimes refer to deterministic matrices A_j as to observables because of the application of local laws to the *eigenstate thermalization hypothesis*).

Definition 2.4 (Regular observables)**.** *Regular part of A with respect to spectral parameters $z_1, z_2 \in \mathbb{C}\backslash\mathbb{R}$:*

$$A^{\circ_{z_1,z_2}} := A - \varphi(z_1, A, z_2) \cdot I,$$

where

$$\varphi(z_1, A, z_2) = u_\delta(z_1, z_2)\frac{\langle M(E_1 + i\Im z_1) AM(E_2 + i\sigma\Im z_2)\rangle}{\langle M(E_1 + i\Im z_1) M(E_2 + i\sigma\Im z_2)\rangle}, \tag{2.9}$$

$$u_\delta(z_1, z_2) = u_\delta(E_1 - E_2)u_\delta(\Im z_1)u_\delta(\Im z_2),$$

where u_δ is a smooth cut-off function with $supp(u) \subset [-2\delta, 2\delta]$, $u_\delta(x) = 1$, $x \in [\delta, \delta]$ and $\sigma = -\operatorname{sign}(\Im z_1 \Im z_2)$. Fixed parameter $\delta > 0$ is N-independent and will be chosen later.

We will say that matrix B is regular with respect to spectral parameters z_1 and z_2 if $B^{\circ_{z_1,z_2}} = B$.

Definition 2.5 (Spectral domain)**.** *Let ρ be the self-consistent density of states corresponding to* (2.3)*. Fix (small) $\kappa, \varepsilon > 0$. The (κ, ε)-spectral domain is the following N-dependent set in the complex plane:*

$$\boldsymbol{D}_{(\kappa,\varepsilon)} := \left\{ z = E + i\eta \in \mathbb{C} : E \in \operatorname{supp}(\rho), dist\,(E, \partial\operatorname{supp}(\rho)) > \kappa, N^{-1+\varepsilon} < |\eta| < N^{100} \right\}.$$

Remark 2.6. *(1) We will refer to $\boldsymbol{D}_{(\kappa,\varepsilon)}$ as to the spectral domain omitting the dependence on κ and ε.*

(2) Further the control parameter η will be the minimum of absolute values of imaginary parts of spectral parameters participating in the formula.

Note that $\|A^{\circ_{z_1,z_2}}\| \lesssim 1$, when z_1, z_2 are in the spectral domain and $\|A\| \lesssim 1$. Now we can formulate our main result.

Theorem 2.7 (Multi-resolvent local laws with 1 and 2 observables)**.** *For spectral parameters in the spectral domain and for bounded observables it holds that*

1.1) 1-G averaged local law:

$$|\langle (G(z) - M(z))A^{\circ_{z,z}}\rangle| \prec \frac{1}{N\sqrt{\eta}},$$

1.2) 2-G isotropic local law:

$$|\langle \boldsymbol{x}, (G(z_1)A^{\circ_{z_1,z_2}}G(z_2) - M(z_1, A^{\circ_{z_1,z_2}}, z_2))\boldsymbol{y}\rangle| \prec \frac{1}{\sqrt{N\eta^2}},$$

2.1) 2-G averaged local law:

$$\left|\left\langle (G(z_1)A_1^{\circ_{z_1,z_2}}G(z_2) - M(z_1, A_1^{\circ_{z_1,z_2}}, z_2))A_2^{\circ_{z_2,z_1}}\right\rangle\right| \prec \frac{1}{N\eta},$$

2.2) 3-G isotropic local law:

$$\left|\left\langle \boldsymbol{x}, \left(G(z_1)A_1^{\circ_{z_1,z_2}}G(z_2)A_2^{\circ_{z_2,z_3}}G(z_3) - M\left(z_1, A_1^{\circ_{z_1,z_2}}, z_2, A_2^{\circ_{z_2,z_3}}, z_3\right)\right)\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\sqrt{N\eta^3}}.$$

We do not present the entire proof of Theorem 2.7, but focus on one of its main ideas (Lemma 3.2) in the following section. For the complete proof we refer to [2, Section 6].

Theorem 2.7 can be used to establish the *Eigenstate Thermalization Hypothesis* (ETH) for deformed Wigner matrices (see [2, Theorem 2.6]), which roughly speaking states that any observable becomes essentially diagonal in the eigenbasis of a deformed Wigner matrix and gives quantitative estimates for this phenomenon. This, in turn, implies after some argument the *equipartition principle* for Wigner matrices [2, Theorem 2.2], which in the simplest setting means that for Wigner matrices W_1 and W_2 and the eigenbasis $\{\boldsymbol{u}_j\}_{j=1}^N$ of $H := W_1 + W_2$ the energy $\lambda_j = \langle \boldsymbol{u}_j, H\boldsymbol{u}_j\rangle$ is equally shared between the summands:

$$\langle \boldsymbol{u}_j, W_1\boldsymbol{u}_j\rangle \approx \frac{\lambda_j}{2} \approx \langle \boldsymbol{u}_j, W_2\boldsymbol{u}_j\rangle.$$

Theorem 2.2 from [2] also gives quantative estimates for fluctuations of $\langle \boldsymbol{u}_j, W_k\boldsymbol{u}_j\rangle$, $k = 1, 2$, around $\lambda_j/2$.

3. Main technical lemma

In order to estimate the differences in Theorem 2.7 we introduce the following control quantities:

$$\Psi_k^{\mathrm{av}}(\boldsymbol{z}_k, \boldsymbol{A}_k) := N\eta^{k/2}|\langle G_1A_1\cdots G_kA_k - M(z_1, A_1, ..., z_k)A_k\rangle| \,, \tag{3.1}$$

$$\Psi_k^{\mathrm{iso}}(\boldsymbol{z}_{k+1}, \boldsymbol{A}_k, \boldsymbol{x}, \boldsymbol{y}) := \sqrt{N\eta^{k+1}}\left|\big(G_1A_1\cdots A_kG_{k+1} - M(z_1, A_1, ..., A_k, z_{k+1})\big)_{\boldsymbol{xy}}\right| \tag{3.2}$$

for $k \in \mathbf{N}$, where we used the short hand notations

$$G_i := G(z_i)\,, \quad \eta := \min_i |\Im z_i|\,, \quad \boldsymbol{z}_k := (z_1, ..., z_k)\,, \quad \boldsymbol{A}_k := (A_1, ..., A_k)\,.$$

The deterministic matrices $\|A_i\| \le 1$, $i \in [k]$, are assumed to be *regular*. For convenience, we extend the above definitions to $k = 0$ by

$$\Psi_0^{\mathrm{av}}(z) := N\eta|\langle G(z) - M(z)\rangle|\,, \quad \Psi_0^{\mathrm{iso}}(z, \boldsymbol{x}, \boldsymbol{y}) := \sqrt{N\eta}\big|\big(G(z) - M(z)\big)_{\boldsymbol{xy}}\big|.$$

We will also use the definition of the *second order renormalization*, denoted by underline, from [3]. For a function $f(W)$ of the Wigner matrix W, we define

$$\underline{Wf(W)} := Wf(W) - \tilde{\mathbb{E}}\left[\tilde{W}\left(\partial_{\tilde{W}}f\right)(W)\right],$$

where $\partial_{\tilde{W}}$ denotes the directional derivative in the direction of $\tilde{W}$, which is a GUE/GOE matrix (depending on the symmetry class of W) that is independent of W. The expectation is taken w.r.t. the matrix $\tilde{W}$. Normally $\underline{Wf(W)}$ is much smaller then $Wf(W)$ in the sense of stochastic domination. We will often use the following representation of resolvent:

$$G = M - M\underline{WG} + M\langle G - M\rangle G. \tag{3.3}$$

Definition 3.1 (Uniform bounds in the spectral domain)**.** *For a fixed $k \in \mathbf{N}$ we say that the bounds*

$$\begin{aligned} &\big|\langle G(z_1)B_1 \;\cdots\; G(z_k)B_k - M(z_1, B_1, ..., z_k)B_k\rangle\big| \prec \mathcal{E}^{\mathrm{av}}\,, \\ &\left|\big(G(z_1)B_1 \;\cdots\; B_kG(z_{k+1}) - M(z_1, B_1, ..., B_k, z_{k+1})\big)_{\boldsymbol{xy}}\right| \prec \mathcal{E}^{\mathrm{iso}} \end{aligned} \tag{3.4}$$

hold uniformly *for some deterministic control parameters $\mathcal{E}^{\mathrm{av/iso}} = \mathcal{E}^{\mathrm{av/iso}}(N, \eta)$, depending only on N and $\eta := \min_i |\Im z_i|$, if the implicit constants in* (3.4) *are uniform in bounded deterministic matrices $\|B_j\| \le 1$, deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \le 1$, and* admissible *spectral parameters $z_j \in \mathbf{D}$ satisfying $1 \ge \eta := \min_j |\Im z_j|$.*

Moreover, we may allow for additional restrictions on the deterministic matrices. For example, we may talk about uniformity under the additional assumption that some (or all) of the matrices are regular.

The following Lemma is one of the main results of the rotational project and is required for the proof of Theorem 2.7.

Lemma 3.2 (Representation as full underlined)**.** *Assume that $\Psi_j^{\mathrm{av/iso}} \prec \psi_j^{\mathrm{av/iso}}$ holds for $1 \le j \le 4$ uniformly in regular matrices. Then we have*

$$\left\langle \left(G(z_1) - M(z_1)\right) A_1^{\circ_{z_1,z_1}} \right\rangle = -\left\langle \underline{WG(z_1)A_1'} \right\rangle + \mathcal{O}_\prec\left(\mathcal{E}_1^{\mathrm{av}}\right), \tag{3.5}$$

$$\left(G(z_1)A_1^{\circ_{z_1,z_2}}G(z_2) - M\left(z_1, A_1^{\circ_{z_1,z_2}}, z_2\right)\right)_{\boldsymbol{xy}} = -\left(\underline{G(z_1)A_1'WG(z_2)}\right)_{\boldsymbol{xy}} + \mathcal{O}_\prec\left(\mathcal{E}_1^{\mathrm{iso}}\right), \tag{3.6}$$

$$\left\langle \left(G(z_1)A_1^{\circ_{z_1,z_2}}G(z_2) - M(z_1, A_1^{\circ_{z_1,z_2}}, z_2)\right) A_2^{\circ_{z_2,z_1}} \right\rangle = -\left\langle \underline{WG(z_1)A_1^{\circ_{z_1,z_2}}G(z_2)A_2'} \right\rangle + \mathcal{O}_\prec\left(\mathcal{E}_2^{\mathrm{av}}\right), \tag{3.7}$$

$$\begin{aligned} &\left(G(z_1)A_1^{\circ_{z_1,z_2}}G(z_2)A_2^{\circ_{z_2,z_3}}G(z_3) - M\left(z_1, A_1^{\circ_{z_1,z_2}}, z_2, A_2^{\circ_{z_2,z_3}}, z_3\right)\right)_{\boldsymbol{xy}} \\ &\qquad = -\left(\underline{G(z_1)A_1'WG(z_2)A_2^{\circ_{z_2,z_3}}G(z_3)}\right)_{\boldsymbol{xy}} + \mathcal{O}_\prec\left(\mathcal{E}_2^{\mathrm{iso}}\right), \end{aligned} \tag{3.8}$$

where

$$\mathcal{E}_1^{\mathrm{av}} = \frac{1}{N\sqrt{\eta}}\left(1 + \frac{\psi_1^{\mathrm{av}}}{N\eta}\right),$$

$$\mathcal{E}_1^{\mathrm{iso}} = \frac{1}{\sqrt{N\eta^2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{N\eta}+\frac{\psi_1^{\mathrm{av}}}{\sqrt{N\eta}}\right),$$
$$\mathcal{E}_2^{\mathrm{av}} = \frac{1}{N\eta}\left(1+\psi_1^{\mathrm{av}}+\frac{(\psi_1^{\mathrm{av}})^2}{N\eta}+\frac{\psi_2^{\mathrm{av}}}{N\eta}\right),$$
$$\mathcal{E}_2^{\mathrm{iso}} = \frac{1}{\sqrt{N\eta^3}}\left(1+\psi_1^{\mathrm{iso}}+\frac{\psi_2^{\mathrm{iso}}}{N\eta}+\frac{\psi_1^{\mathrm{iso}}\psi_1^{\mathrm{av}}}{N\eta}\right).$$

and matrices A_1', A_2' (which may be different for different $\psi_j^{\mathrm{av/iso}}$) are regular in the corresponding product chains and depend linearly on the matrices participating in the LHS of (3.5) - (3.8).

In the two following subsections we prove (3.5) and (3.6), while the proof of (3.7) and (3.8) is ideologically analogous modulo technical details.

3.1. **Proof of** (3.5). For brevity we will sometimes omit the argument z_1 in $G = G(z_1)$ and $M = M(z_1)$ in this subsection. In the proof of (3.5) only regular parts with respect to the equal spectral parameters appear, so one can forget for a moment about the characteristic function in Definition 2.4. The regularity condition for A_1 now reads as $\langle A_1 \operatorname{Im} M\rangle = 0$.

Define the following one-body stability operator which acts on $N \times N$ matrices:

$$\mathcal{B} := 1 - M(z_1)\langle\cdot\rangle M(z_1).$$

With this notation (3.3) can be rewritten as follows

$$\mathcal{B}\left[G-M\right] = -M\underline{WG} + M\langle G-M\rangle(G-M). \tag{3.9}$$

Direct computation shows that $\mathcal{B}$ is invertible and gives the explicit formula for $\mathcal{B}^{-1}$:

$$\mathcal{B}^{-1} = 1 + \frac{\langle\cdot\rangle}{1-\langle M^2\rangle}M^2. \tag{3.10}$$

After inverting $\mathcal{B}$ in (3.9), multiplying the obtained identity by A_1 and taking the trace we get

$$\langle (G-M)A_1\rangle = -\langle \mathcal{B}^{-1}\left[M\underline{WG}\right]A_1\rangle + \langle G-M\rangle\langle\mathcal{B}^{-1}\left[M(G-M)\right]A_1\rangle \tag{3.11}$$

Now we want to throw the action of $\mathcal{B}^{-1}$ to the matrix A_1. Note that for arbitrary $N\times N$ matrices R_1 and R_2 it holds that

$$\langle\mathcal{B}^{-1}\left[R_1\right]R_2\rangle = \langle\mathcal{B}^{-1}\left[R_1\right](R_2^*)^*\rangle = \left\langle R_1\left(\left(\mathcal{B}^{-1}\right)^*\left[R_2^*\right]\right)^*\right\rangle = \langle R_1\mathcal{X}_{11}\left[R_2\right]\rangle,$$

where $\mathcal{X}_{11}[R] := \left(\left(\mathcal{B}^{-1}\right)^*\left[R^*\right]\right)^*$ is a linear operator on the space of $N\times N$ matrices equipped with the scalar product $(R_1, R_2) := \langle R_1 R_2^*\rangle$. It is easy to find $\mathcal{X}_{11}$ explicitly:

$$\mathcal{X}_{11}[R] = R + \frac{\langle MRM\rangle}{1-\langle M^2\rangle}.$$

An important feature of $\mathcal{X}_{11}$ is that $\|\mathcal{X}_{11}[R]\| \lesssim 1$ for (z_1,z_1)-regular R with $\|R\|\lesssim 1$. From (3.11) it follows that

$$\langle (G-M)A_1\rangle = -\langle\underline{WG}\left(\mathcal{X}_{11}\left[A_1\right]M\right)\rangle + \langle G-M\rangle\langle(G-M)\left(\mathcal{X}_{11}[A_1]M\right)\rangle.$$

Decompose $\mathcal{X}[A_1]M$ into the regular part and a multiple of identity:

$$\begin{aligned}\langle (G-M)A_1\rangle &= -\langle\underline{WG}\left(\mathcal{X}_{11}\left[A_1\right]M\right)^\circ\rangle + \langle G-M\rangle\langle(G-M)\left(\mathcal{X}_{11}[A_1]M\right)^\circ\rangle\\ &\quad + \varphi\left(\mathcal{X}_{11}[A_1]M\right)\left(\langle G-M\rangle^2 - \langle\underline{WG}\rangle\right).\end{aligned} \tag{3.12}$$

Denote $A_1' := \left(\mathcal{X}_{11}\left[A_1\right]M\right)^\circ$. Our aim is to obtain (3.5) from (3.12) by estimating the second and the third terms in the RHS of (3.12). From the usual averaged local law and the definition of ψ_1^{av} it follows that

$$\langle G-M\rangle\langle(G-M)\left(\mathcal{X}_{11}[A_1]M\right)^\circ\rangle = \mathcal{O}_\prec\left(\frac{1}{N\eta}\cdot\frac{\psi_1^{\mathrm{av}}}{N\sqrt{\eta}}\right).$$

Since z_1 is in the spectral domain, the denominator of $\varphi\left(\mathcal{X}_{11}[A_1]M\right)$ is bounded away from zero and

$$\left|\varphi\left(\mathcal{X}_{11}[A_1]M\right)\right| \lesssim \left|\langle\left(\mathcal{X}_{11}[A_1]M\right)\Im M\rangle\right| = \left|\langle A_1\mathcal{B}^{-1}\left[M\Im M\right]\rangle\right|.$$

Calculate $\mathcal{B}^{-1}[M\Im M]$ using (3.10) and (2.3):

$$\begin{aligned}\mathcal{B}^{-1}[M\Im M] &= \frac{\mathcal{B}^{-1}\left[M^2 - MM^*\right]}{2i} = \frac{M^2 - MM^*}{2i} + \frac{1}{2i}\frac{\langle M^2 - MM^*\rangle}{1-\langle M^2\rangle}M^2 \\ &= \frac{i}{2}\frac{\Im M}{\eta + \langle \Im M\rangle} + \frac{1}{2i}\frac{1-\langle MM^*\rangle}{1-\langle M^2\rangle}M^2 = \frac{i}{2}\frac{\Im M}{\eta+\langle\Im M\rangle} + \mathcal{O}(\eta)M^2,\end{aligned} \tag{3.13}$$

where we also used that

$$MM^* = \frac{\Im M}{\eta + \langle \Im M\rangle} \quad \text{and} \quad 1 - \langle MM^*\rangle = \frac{\eta}{\eta+\langle\Im M\rangle} = \mathcal{O}(\eta).$$

We get an upper bound for $\varphi(\mathcal{X}_{11}[A_1]M)$ by multiplying (3.13) by A_1, taking trace and recalling that A_1 is (z_1, z_1)-regular:

$$|\varphi(\mathcal{X}_{11}[A_1]M)| \lesssim \left|\frac{i}{2}\frac{\langle A_1 \Im M\rangle}{\eta+\langle \Im M\rangle} + \mathcal{O}(\eta)\langle A_1 M^2\rangle\right| = \mathcal{O}(\eta).$$

It is left to deal with the term $\langle G - M\rangle^2 - \langle \underline{WG}\rangle$. We do this by rewriting the term in the following way:

$$\begin{aligned}&\langle G-M\rangle^2 - \langle\underline{WG}\rangle \\ &= \langle G-M\rangle^2 - \left\langle -\frac{1}{M}(G-M) + \langle G-M\rangle(G-M) + \langle G-M\rangle M\right\rangle = \left\langle (G-M)\left(\frac{1}{M} - \langle M\rangle\right)\right\rangle.\end{aligned}$$

So, the usual averaged local law implies that

$$\left|\langle G-M\rangle^2 - \langle\underline{WG}\rangle\right| \prec \frac{1}{N\eta}.$$

Collecting the obtained estimates of terms in (3.12) we get (3.5).

3.2. **Proof of** (3.6)**.** We will use the shorthand notations $G_j := G(z_j)$, $M_j := M(z_j)$ for $j = 1, 2$. Consider the product $G_1\tilde{A}G_2$ for an arbitrary $N\times N$ matrix $\tilde{A}$ and use (3.3) for G_2:

$$G_1\tilde{A}G_2 = M_1\tilde{A}M_2 + (G_1 - M_1)\tilde{A}M_2 - G_1\tilde{A}M_2\underline{WG_2} + \langle G_2 - M_2\rangle G_1\tilde{A}M_2G_2. \tag{3.14}$$

Extend the underline on the entire third term:

$$\underline{G_1\tilde{A}M_2WG_2} = G_1\tilde{A}M_2\underline{WG_2} + \mathbb{E}_{\tilde{W}}\left[G_1\tilde{W}G_2\tilde{A}M_2\tilde{W}G_2\right] = G_1\tilde{A}M_2\underline{WG_2} + \langle G_1\tilde{A}M_2\rangle G_1G_2. \tag{3.15}$$

Plugging (3.15) into (3.14) and rearranging the summands we get that

$$\begin{aligned}&G_1\left(\tilde{A} - \langle M_1\tilde{A}M_2\rangle\right)G_2 \\ &= M_1\tilde{A}M_2 + (G_1 - M_1)\tilde{A}M_2 - \underline{G_1\tilde{A}M_2WG_2} + \langle G_2 - M_2\rangle G_1\tilde{A}M_2G_2 + \langle (G_1-M_1)\tilde{A}M_2\rangle G_1G_2.\end{aligned} \tag{3.16}$$

Now for a (z_1, z_2)-regular matrix A we chose the matrix $\tilde{A}$ in such a way that $A = \tilde{A} - \langle M_1\tilde{A}M_2\rangle$. Note that such choice of $\tilde{A}$ exists and is unique:

$$\tilde{A} = \mathcal{X}_{12}[A] := A + \frac{\langle M_1AM_2\rangle}{1 - \langle M_1M_2\rangle}.$$

Explicit formulas for $\mathcal{X}_{12}$ and $M(z_1, A, z_2)$ give that $M_1\mathcal{X}_{12}[A]M_2 = M(z_1, A, z_2)$. Decompose the matrix $\mathcal{X}_{12}[A]M_2$ in the last three terms in the RHS of (3.16) with respect to spectral parameters z_1 and z_2:

$$\begin{aligned}G_1AG_2 &= M(z_1, A, z_2) + (G_1 - M_1)\mathcal{X}_{12}[A]M_2 - \underline{G_1\left(\mathcal{X}_{12}[A]M_2\right)^{\circ_{12}}WG_2} \\ &+ \langle G_2 - M_2\rangle G_1\left(\mathcal{X}_{12}[A]M_2\right)^{\circ_{12}}G_2 + \left\langle (G_1-M_1)\left(\mathcal{X}_{12}[A]M_2\right)^{\circ_{12}}\right\rangle G_1G_2 \\ &+ \varphi(z_1, \mathcal{X}_{12}[A]M_2, z_2)\left\{-\underline{G_1WG_2} + \langle G_2 - M_2\rangle G_1G_2 + \langle G_1 - M_1\rangle G_1G_2\right\}.\end{aligned} \tag{3.17}$$

The coefficient by $\varphi(z_1, \mathcal{X}_{12}[A]M_2, z_2)$ equals to

$$-G_1 + G_1\left(\frac{1}{M_2} - \langle M_1\rangle\right)G_2,$$

this can be seen by expanding the underline in $\underline{G_1WG_2}$ by definition. Denote $\Phi := M_2^{-1} - \langle M_1\rangle$. Easy computation shows that $M(z_1, \Phi, z_2) = M_1$. This yields that the term $\{\cdots\}$ in (3.17) equals to

$$\begin{aligned}&-(G_1 - M_1) + (G_1\Phi G_2 - M(z_1, \Phi, z_2))\\ &\quad = -(G_1 - M_1) + (G_1\Phi^{\circ_{12}}G_2 - M((z_1, \Phi^{\circ_{12}}, z_2)) + \varphi(z_1, \Phi, z_2)(G_1G_2 - M(z_1, I, z_2)).\end{aligned}$$

Next we substitute $A := \Phi^{\circ_{12}}$ into (3.17), interpret the identity as a linear equation with unkown variable $G_1\Phi^{\circ_{12}}G_2 - M(z_1, \Phi^{\circ_{12}}, z_2)$ and find this variable. By plugging the result into (3.17) and substituting $A := A_1$ we get the identity which is the starting point for estimating the error term in (3.6):

(3.18)
$$\begin{aligned}G_1A_1G_2 - M(z_1, A_1, z_2) &= (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1(\mathcal{X}_{12}[A_1]M_2)^{\circ_{12}}WG_2}\\ &+ \langle G_2 - M_2\rangle G_1(\mathcal{X}_{12}[A_1]M_2)^{\circ_{12}}G_2 + \langle (G_1 - M_1)(\mathcal{X}_{12}[A_1]M_2)^{\circ_{12}}\rangle G_1G_2\\ &+ \frac{\varphi(z_1, \mathcal{X}_{12}[A_1]M_2, z_2)}{1 - \varphi(z_1, \mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2, z_2)}\Bigg((G_1 - M_1)\mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2\\ &- \underline{G_1(\mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2)^{\circ_{12}}WG_2} + \langle G_2 - M_2\rangle G_1(\mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2)^{\circ_{12}}G_2\\ &+ \langle (G_1 - M_1)(\mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2)^{\circ_{12}}\rangle G_1G_2 - (G_1 - M_1) + \varphi(z_1, \Phi, z_2)(G_1G_2 - M(z_1, I, z_2))\Bigg).\end{aligned}$$

At first we need to show that $1 - \varphi(z_1, \mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2, z_2)$ does not vanish.

Lemma 3.3. *For small enough $\delta > 0$ we have that*

$$\left|\frac{1}{1 - \varphi(z_1, \mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2, z_2)}\right| \lesssim 1.$$

Proof of Lemma 3.3: If $u_\delta(z_1, z_2) = 0$, then $1 - \varphi(...) = 1$. Otherwise $z_1 \approx z_2$ or $z_1 \approx \bar{z}_2$. In this regime we only deal with the extreme case $u_\delta = 1$. For the intermediate case $u_\delta \in (0, 1)$ some minor additional ideas are required, however, we do not want to discuss them here. We consider two cases: when z_1, z_2 are in different half-planes and when they are in the same one.

(1) $\underline{\Im z_1 \cdot \Im z_2 < 0}$. We compute the fraction which we need to estimate using the definitions of Φ, regular part and $\mathcal{X}_{12}$:

$$(1 - \varphi(z_1, \mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2, z_2))^{-1} = \left(1 - \frac{\langle M_1M_2\rangle}{\langle M_1M_2\rangle} + \frac{\langle M_1\rangle}{\langle M_1M_2\rangle}\cdot\frac{\langle M_1M_2\rangle}{\langle M_1M_2^2\rangle}\right)^{-1} = \frac{\langle M_1M_2\rangle^2}{\langle M_1\rangle\langle M_1M_2^2\rangle}$$

Since $\langle M_1M_2\rangle^2 \lesssim 1$ and $|\langle M_1\rangle| \sim 1$, we only need to show that $|\langle M_1M_2^2\rangle|^{-1} \lesssim 1$. The following calculation establishes this inequality:

$$\begin{aligned}\frac{1}{|\langle M_1M_2^2\rangle|} &= \frac{|z_1 - z_2 + \langle M_1\rangle - \langle M_2\rangle|}{|\langle (M_1 - M_2)M_2\rangle|} \lesssim \frac{1}{|\langle (M_1 - M_2)M_2\rangle|}\\ &= \frac{1}{|\langle (M_1 - M_1^*)M_2\rangle + \langle (M_1^* - M_2)M_2\rangle|}\\ &= \frac{1}{|\langle (M_1 - M_1^*)M_1^*\rangle + \langle (M_1 - M_1^*)(M_2 - M_1^*)\rangle + \langle (M_1^* - M_2)M_2\rangle|}\\ &= \frac{1}{|2i\langle \Im M_1\cdot M_1^*\rangle + 2i\langle \Im M_1\cdot M'(\zeta_1)(z_2 - \bar{z_1})\rangle + \langle M'(\zeta_2)(\bar{z_1} - z_2)M_2\rangle|}\\ &= \frac{1}{\left|2\langle (\Im M_1)^2\rangle + 2i\langle \Im M_1\Re M_1\rangle + \mathcal{O}(|z_1 - \bar{z}_2|)\right|}\\ &\lesssim \frac{1}{\left|(\langle \Im M_1)^2\rangle\right| + \mathcal{O}(|z_1 - \bar{z}_2|)} \le \frac{1}{\langle \Im M_1\rangle^2 + \mathcal{O}(|z_1 - \bar{z}_2|)} \lesssim 1.\end{aligned}$$

(2) $\Im z_1 \cdot \Im z_2 > 0$. In this case we can also compute that

$$\frac{1}{1-\varphi(z_1, \mathcal{X}_{12}\left[\Phi^{\circ_{12}}\right] M_2, z_2)} = \frac{\langle M_1 M_2^* \rangle^2 (1 - \langle M_1 M_2 \rangle)}{\langle M_1 M_2 M_2^* \rangle (\langle M_1 \rangle \langle M_1 M_2^* \rangle - \langle M_1 M_2^{-1} M_2^* \rangle)}$$

We give lower estimates of order 1 for each factor in the denominator:

$$\begin{aligned}
|\langle M_1 M_2 M_2^* \rangle| &= \left|\left\langle M_1^2 M_1^* \right\rangle\right| + \mathcal{O}(|z_1 - z_2|) = \frac{|\langle M_1 \Im M_1 \rangle|}{|\eta + \langle \Im M_1 \rangle|} + \mathcal{O}(|z_1 - z_2|) \\
&\gtrsim |\langle M_1 \Im M_1 \rangle| + \mathcal{O}(|z_1 - z_2|) = \left| \langle \Re M_1 \cdot \Im M_1 \rangle + i \left\langle (\Im M_1)^2 \right\rangle \right| + \mathcal{O}(|z_1 - z_2|) \\
&\geq \left| \left\langle (\Im M_1)^2 \right\rangle \right| + \mathcal{O}(|z_1 - z_2|) \geq \langle \Im M_1 \rangle^2 + \mathcal{O}(|z_1 - z_2|) \gtrsim 1.
\end{aligned}$$

And for the second factor:

$$\begin{aligned}
|\langle M_1 \rangle \langle M_1 M_2^* \rangle - \langle M_1 M_2 M_2^* \rangle| &= \left| \langle M_1 \rangle \langle M_1 M_1^* \rangle - \left\langle M_1 M_1^{-1} M_1^* \right\rangle \right| + \mathcal{O}(|z_1 - z_2|) \\
&= |(\langle M_1 \rangle - \langle M_1^* \rangle) + \langle M_1 \rangle (\langle M_1 M_1^* \rangle - 1)| + \mathcal{O}(|z_1 - z_2|) \\
&\gtrsim |\langle \Im M_1 \rangle| - |1 - \langle M_1 M_1^* \rangle| + \mathcal{O}(|z_1 - z_2|) = |\langle \Im M_1 \rangle| + \mathcal{O}(\eta + |z_1 - z_2|) \gtrsim 1.
\end{aligned}$$

This finishes the proof of Lemma 3.3 in the second case. □

Next, we take scalar product of (3.18) with two deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$ satisfying $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \leq 1$. In the resulting expression we need to discuss separately two terms:

$$\left(\left\langle (G_1 - M_1) \left(\mathcal{X}_{12}[\Phi^{\circ_{12}}] M_2 \right)^{\circ_{12}} \right\rangle G_1 G_2 \right)_{\boldsymbol{xy}} \tag{3.19}$$

and

$$\left(\varphi\left(z_1, \Phi, z_2\right) \left(G_1 G_2 - M(z_1, I, z_2)\right) \right)_{\boldsymbol{xy}}. \tag{3.20}$$

Estimating (3.19)**.** In the matrix product under the trace in (3.19) the matrix $\left(\mathcal{X}_{12}[\Phi^{\circ_{12}}] M_2\right)^{\circ_{12}}$ may not be (z_1, z_1)-regular. However, the following continuity of regular part with respect to spectral parameters holds:

Lemma 3.4. *Let R be a deterministic matrix with $\|R\| \leq 1$, z_1, z_2 are in the spectral domain, $E_j = \Re z_j$, $\eta_j = |\Im z_j|$, $j = 1, 2$. Then in holds that*

$$R^{\circ_{z_1, z_2}} = R^{\circ_{z_1, z_1}} + \mathcal{O}\left(|E_1 - E_2| + |\eta_1 - \eta_2|\right),$$

where implicit constant in $\mathcal{O}$ does not depend on R, z_1, z_2, N, but may depend on δ.

The proof of Lemma 3.4 is an easy application of (2.3) and hence is omitted. This lemma gives that (3.19) equals to

$$\left(\left\langle (G_1 - M_1) \left(\mathcal{X}_{12}[\Phi^{\circ_{12}}] M_2 \right)^{\circ_{11}} \right\rangle + \langle G_1 - M_1 \rangle \mathcal{O}\left(|E_1 - E_2| + |\eta_1 - \eta_2|\right) \right) (G_1 G_2)_{\boldsymbol{xy}}. \tag{3.21}$$

In the case $\Im z_1 \Im z_2 < 0$ we use resolvent identity in order to estimate the second factor:

$$\begin{aligned}
\left| (G_1 G_2)_{\boldsymbol{xy}} \right| &= \left| \frac{(G_1)_{\boldsymbol{xy}} - (G_2)_{\boldsymbol{xy}}}{z_1 - z_2} \right| \prec \left(1 + \frac{1}{\sqrt{N\eta_1}} + \frac{1}{\sqrt{N\eta_2}} \right) \cdot \frac{1}{|E_1 - E_2| + \eta_1 + \eta_2} \\
&\lesssim \frac{1}{|E_1 - E_2| + \eta_1 + \eta_2},
\end{aligned} \tag{3.22}$$

where we used usual isotropic local law. In the case $\Im z_1 \Im z_2 > 0$ we employ the integral representation in this term:

$$(G_1 G_2)_{\boldsymbol{xy}} = (G(E_1 + i\eta_1) G(E_2 + i\eta_2))_{\boldsymbol{xy}} = \frac{1}{2\pi i} \int\limits_{\mathbb{R}} \frac{(G(x + i\eta/3))_{\boldsymbol{xy}}}{(x + i\eta/3 - z_1)(x + i\eta/3 - z_2)} dx.$$

Again by using the usual isotropic local law we get that

$$\begin{aligned}\left|(G_1G_2)_{\boldsymbol{xy}}\right| &\prec \left(1+\frac{1}{\sqrt{N\eta}}\right)\int\limits_{\mathbb{R}} \frac{dx}{|x-E_1-i(\eta_1-\eta/3)|\cdot|x-E_2-i(\eta_2-\eta/3)|} \\ &\prec \left(1+\frac{1}{\sqrt{N\eta}}\right)\frac{\log N}{|E_1-E_2|+\eta_1+\eta_2} \prec \frac{1}{|E_1-E_2|+\eta_1+\eta_2}.\end{aligned} \tag{3.23}$$

where $\log N$ was absorbed by $\prec$. Now we use either (3.22) or (3.23) depending on sign of $\Im z_1\Im z_2$ in (3.21) and get the estimate for (3.19):

$$\begin{aligned}|(3.19)| &\prec \frac{\left|\left\langle (G_1-M_1)\left(\mathcal{X}_{12}[\Phi^{\circ_{12}}]M_2\right)^{\circ_{11}}\right\rangle\right| + |\langle G_1-M_1\rangle|\,\mathcal{O}(|E_1-E_2|+|\eta_1-\eta_2|)}{|E_1-E_2|+\eta_1+\eta_2} \\ &\prec \frac{1}{|E_1-E_2|+\eta_1+\eta_2}\cdot\frac{\psi_1^{\mathrm{av}}}{N\sqrt{\eta}} + \frac{1}{N\eta} \prec \frac{1}{N\eta} + \frac{1}{\sqrt{N\eta}}\cdot\frac{\psi_1^{\mathrm{av}}}{\sqrt{N\eta}}.\end{aligned}$$

Estimating (3.20)**.** We again distinguish between cases $\Im z_1\Im z_2<0$ and $\Im z_1\Im z_2>0$.

(1) $\Im z_1\Im z_2<0$. For the first factor in (3.20) we have

$$\begin{aligned}|\varphi(z_1,\Phi,z_2)| &= \left|\frac{\langle M_1\Phi M_2\rangle}{\langle M_1M_2\rangle}\right| = \left|\frac{\langle M_1\rangle-\langle M_1\rangle\langle M_1M_2\rangle}{\langle M_1M_2\rangle}\right| \lesssim |1-\langle M_1M_2\rangle| \\ &= \frac{|z_1-z_2|}{|\langle M_1\rangle-\langle M_2\rangle+z_1-z_2|} \lesssim |z_1-z_2| = \mathcal{O}(|E_1-E_2|+\eta_1+\eta_2).\end{aligned} \tag{3.24}$$

In the factor containing G_1G_2 we use resolvent identity and the corresponding property of the deterministic approximation:

$$\begin{aligned}\left|(G_1G_2-M(z_1,I,z_2))_{\boldsymbol{xy}}\right| &= \left|\frac{(G_1-M_1)_{\boldsymbol{xy}}-(G_2-M_2)_{\boldsymbol{xy}}}{z_1-z_2}\right| \\ &\prec \left(\frac{1}{\sqrt{N\eta_1}}+\frac{1}{\sqrt{N\eta_2}}\right)\cdot\frac{1}{|E_1-E_2|+\eta_1+\eta_2} \prec \frac{1}{\sqrt{N\eta}}\cdot\frac{1}{|E_1-E_2|+\eta_1+\eta_2}.\end{aligned} \tag{3.25}$$

Combining bounds (3.24) and (3.25) we get that $|(3.20)|\prec 1/\sqrt{N\eta}$.

(1) $\Im z_1\Im z_2>0$. Using integral representation for G_1G_2 and the same one for $M(z_1,I,z_2)$ we get inequality (3.25) in the given case. This step is fully analogous to what was done during estimating (3.19). For $|\varphi(z_1,\Phi,z_2)|$ we have an obvious upper estimate of order 1 and it is not possible to improve it. But (3.20) comes with the factor $\varphi(z_1,\mathcal{X}[B]M_2,z_2)$, which can be bounded is a nicer way (then just by constant) in the current case:

$$\begin{aligned}|\varphi(z_1,\mathcal{X}[A_1]M_2,z_2)| &= \left|\frac{\langle M_1\mathcal{X}[A_1]M_2M_2^*\rangle}{M_1M_2^*}\right| \lesssim |\langle M_1\mathcal{X}[A_1]M_2M_2^*\rangle| \\ &= \left|\langle M_1A_1M_2M_2^*\rangle + \frac{\langle M_1A_1M_2\rangle}{1-\langle M_1M_2\rangle}\langle M_1M_2M_2^*\rangle\right| \\ &= \left|\frac{1}{\Im z_2+\langle\Im M_2\rangle}\left(\langle M_1A_1\Im M_2\rangle + \frac{\langle M_1A_1M_2\rangle}{1-\langle M_1M_2\rangle}\langle M_1\Im M_2\rangle\right)\right| \\ &\lesssim \left|\frac{1}{2i}\left(\langle M_1A_1M_2\rangle-\langle M_1A_1M_2^*\rangle + \frac{\langle M_1A_1M_2\rangle}{1-\langle M_1M_2\rangle}(\langle M_1M_2\rangle-\langle M_1M_2^*\rangle\right)\right| \\ &\lesssim \left|\frac{\langle M_1A_1M_2\rangle(1-\langle M_1M_2\rangle+\langle M_1M_2\rangle-\langle M_1M_2^*\rangle)}{1-\langle M_1M_2\rangle}\right| \lesssim |1-\langle M_1M_2^*\rangle| \\ &\lesssim |z_1-\bar{z}_2| = \mathcal{O}(|E_1-E_2|+\eta_1+\eta_2).\end{aligned}$$

Therefore,

$$|\varphi(z_1,\mathcal{X}[B]M_2,z_2)\cdot(3.20)| \prec \frac{1}{\sqrt{N\eta}}.$$

Collecting all bounds for the terms in (3.18) we get that (3.6) holds with:

$$A_1' = (\mathcal{X}_{12}[A_1]M_2)^{\circ_{12}} + \frac{\varphi\left(z_1, \mathcal{X}_{12}[A_1]M_2, z_2\right)}{1 - \varphi(z_1, \mathcal{X}_{12}\left[\Phi^{\circ_{12}}\right]M_2, z_2)} \left(\mathcal{X}_{12}\left[\Phi^{\circ_{12}}\right]M_2\right)^{\circ_{12}}.$$

Appendix A. Properties of the deterministic approximation

Definition A.1. *Let $k \in \mathbb{N}$. A pair is a tuple $p = (i, j)$, where $i, j \in \{1, \ldots, k\}$ and $i < j$. All sets of pairs which we consider consist of pairwise distinct pairs (but some of pairs may have common elements). We will use P to denote sets of pairs. $\mathcal{P}_k$ is the collection of all sets of pairs on the set $\{1, \ldots, k\}$.*

Consider $P \in \mathcal{P}_k$. We will say that P is a non-crossing set of pairs if for all $(i_1, j_1), (i_2, j_2) \in P$ such that $i_1 < i_2 < j_1$ it holds that $i_1 < j_2 < j_1$. Denote the collection of all non-crossing sets of pairs on the set $\{1, \ldots, k\}$ by $\mathcal{P}_k^{nc}$.

Definition A.2. *Consider $P \in \mathcal{P}_k^{nc}$. The $N \times N$ matrix $M_P(z_1, A_1, z_2, \ldots, z_k)$ is defined in the following way. We start with the product $M_1A_1M_2A_2\ldots A_{k-1}M_k$. For every $(i, j) \in P$ we cross out of this product the following part: $A_iM_{i+1}\ldots A_{j-1}$. Then $M_P(z_1, A_1, z_2, \ldots, z_k)$ is the string obtained after all "crossings" are completed.*

Note that for each $P \in \mathcal{P}_k^{nc}$ the first factor of $M_P(z_1, A_1, z_2, \ldots, z_k)$ is M_1 and the last is M_k.

Definition A.3. *Consider $P \in \mathcal{P}_k^{nc}$ and $(i, j) \in P$. Consider the product $M_iA_iM_{i+1}\ldots M_j$. For all $(s, t) \in P$ such that $i \le s < t \le j$ and $(s, t) \neq (i, j)$ we cross $A_sM_{s+1}\ldots A_{t-1}$ out of this product. Then $\tilde{m}_P^{(i,j)}(z_1, A_1, z_2, \ldots, z_k)$ is the normalized trace of the obtained product. We also denote*

$$m_P^{(i,j)}(z_1, A_1, z_2, \ldots, z_k) := \frac{\tilde{m}_P^{(i,j)}(z_1, A_1, z_2, \ldots, z_k)}{1 - \langle M_iM_j\rangle}$$

and

$$m_P(z_1, A_1, z_2, \ldots, z_k) := \prod_{(i,j)\in P} m_P^{(i,j)}(z_1, A_1, z_2, \ldots, z_k).$$

Theorem A.4.

$$M(z_1, A_1, z_2, \ldots, z_k) = \sum_{P\in\mathcal{P}_k^{nc}} m_P(z_1, A_1, z_2, \ldots, z_k)M_P(z_1, A_1, z_2, \ldots, z_k). \tag{A.1}$$

Proof of Theorem A.4: It is sufficient to check that

$$\tilde{M}(z_1, A_1, z_2, \ldots, z_k) := \sum_{P\in\mathcal{P}_k^{nc}} m_PM_P$$

satisfies recursive formula (2.7). We start with analyzing $\left\langle \tilde{M}(z_1, A_1, z_2, \ldots, z_l)\right\rangle$ in terms of $\mathcal{P}_l^{nc}$:

$$\left\langle \tilde{M}(z_1, A_1, z_2, \ldots, z_l)\right\rangle = \sum_{P\in\mathcal{P}_l^{nc}} m_P\langle M_P\rangle = \sum_1 m_P\langle M_P\rangle + \sum_2 m_P\langle M_P\rangle,$$

where the first sum goes over all non-crossing sets of pairs which do not contain the pair $(1, l)$, and the second over the rest of sets of non-crossing pairs. There is a natural bijection between this two sets: if P belongs to the first group of indices, then $P \cup (1, l)$ is in the second group. Also each set from the second group of indices is of the form $P \cup (1, l)$, where P is from the first group. Thus we have:

$$\left\langle \tilde{M}(z_1, A_1, z_2, \ldots, z_l)\right\rangle = \sum_1 \left(m_P\langle M_P\rangle + m_{P\cup(1,l)}\left\langle M_{P\cup(1,l)}\right\rangle\right).$$

Note that $M_{P\cup(1,l)} = M_1M_l$. We also have that

$$m_{P\cup(1,l)} = m_P\frac{\langle M_P\rangle}{1 - \langle M_1M_l\rangle}.$$

Therefore,

$$m_P\langle M_P\rangle + m_{P\cup(1,l)}\langle M_{P\cup(1,l)}\rangle = m_P\langle M_P\rangle + m_P\frac{\langle M_P\rangle}{1 - \langle M_1M_l\rangle}\langle M_1M_l\rangle = \frac{m_P\langle M_P\rangle}{1 - \langle M_1M_l\rangle},$$

$$\langle \tilde{M}(z_1, A_1, z_2, \dots, z_l) \rangle M_1 M_l = \sum_{P \in \mathcal{P}_l^{nc}, (1,l) \in P} m_P M_P,$$

$$M_1 A_1 \tilde{M}\,(z_2, A_2, \dots, A_{k-1}, z_k) = \sum_{P \in \mathcal{P}^{(1)}} m_P M_P,$$

where $\mathcal{P}^{(1)}$ is the subset of $\mathcal{P}_k^{nc}$ consisting of all sets of pairs which do not contain 1;

$$\langle \tilde{M}(z_1, A_1, \dots, A_{j-1}, z_j) \rangle M_1 \tilde{M}(z_j, A_j, \dots, A_{k-1}, z_k) = \sum_{P \in \mathcal{P}^{(j)}} m_P M_P,$$

where $\mathcal{P}^{(j)}$ is the collection of all non-crossing sets of pairs containing $(1, j)$ and where $(1, j)$ is the biggest by inclusion of intervals pair which contains 1. Hence $\tilde{M}$ satisfies (2.7). □

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