Central Limit Theorem for Mesoscopic Linear Eigenvalue Statistics of Wigner-type Matrices

Rotation Project in Erdös Group

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January 27, 2022

1 Introduction

The main objective of this project was to learn and understand the main tools and technical details of Theorem 2.2 in [1] and attempt to generalize its proof for Wigner-type matrices. Such generalization was carried out in [2] for large scales, but we conjecture that the result can be improved to the optimal scale by using only resolvent comparison methods. For the sake of simplicity we resort to studying the real symmetric case.

2 Model

2.1 Wigner-type matrices

Definition 2.1. *Let $H = (H_{ij})_{i,j=1}^N$ be an $N \times N$ matrix with independent entries up to real symmetry condition $H = H^t$ satisfying*

$$\mathbb{E}\left[H_{ij}\right] = 0. \tag{2.1}$$

Denote by S the matrix of variances: $S_{ij} := \mathbb{E}\left[|H_{ij}|^2\right]$, which satisfies

$$\frac{c_{sup}}{N} \leq S_{ij} \leq \frac{C_{inf}}{N}, \tag{A}$$

for all $i, j \in \{1, \ldots, N\}$ and some strictly positive constants C_{sup}, c_{inf}.
We assume a uniform bound on all other moments of $\sqrt{N}H_{ij}$, that is for any $k \in \mathbb{N}$ there exists a positive constant C_k independent of N such that

$$\mathbb{E}\left[|\sqrt{N}H_{ij}|^k\right] \leq C_k \tag{2.2}$$

holds for all $i, j \in \{1, \ldots, N\}$.

Additionally, we assume that the matrix of variances satisfies Hölder regularity condition, i.e.

$$|S_{ij} - S_{ij'}| \leq \frac{L}{N}\left(\frac{|j-j'|}{N}\right)^{1/2}, \tag{B}$$

for all $i, j, j' \in \{1, \ldots, N\}$ and some positive constant L.

2.2 Notations

For a vector $x \in \mathbb{C}^N$ we use the standart definitions of ℓ^2 and ℓ^∞ norms, namely,

$$\|x\|_2 = \left(\sum_{j=1}^N |x_j|^2\right)^{1/2}, \quad \|x\|_\infty = \max_j |x_j|.$$

For a linear operator $T : \mathbb{C}^N \to \mathbb{C}^N$ we denote its matrix norms induced by ℓ^2 and ℓ^∞ norms respectively as

$$\|T\|_{\ell^2\to\ell^2} = \sup_{\|x\|_2=1} \|Tx\|_2\,, \quad \|T\|_{\ell^\infty\to\ell^\infty} = \sup_{\|x\|_\infty=1} \|Tx\|_\infty\,.$$

For two vectors $x, y \in \mathbb{C}^N$ we use the angle bracket to denote the ℓ^2 scalar product

$$\langle x, y\rangle = \sum_{j=1}^{N} \bar{x}_j y_j.$$

We use xy to denote a coordinate-wise multiplication of the vectors,

$$(xy)_j = x_j y_j, \quad j \in \{1, \ldots, N\}.$$

Similarly, for a given vector x with non-zero entries, $\frac{1}{x}$ denotes a coordinate-wise multiplicative inverse

$$\left(\frac{1}{x}\right)_j = \frac{1}{x_j}, \quad j \in \{1, \ldots, N\}$$

We will use the following definition of stochastic domination (Definition 6.4 in [3])

Definition 2.2. *Let $X = X^{(N)}(u)$ and $Y = Y^{(N)}(u)$ be two families of random variables depending on a parameter $u \in U^{(N)}$. We say that Y stochastically dominates X uniformly in u if for any $\varepsilon > 0$ and $D > 0$ there exists $N_0(\varepsilon, D)$ such that for any $N \geq N_0(\varepsilon, D)$,*

$$\sup_{u\in U^{(N)}} \mathbb{P}\left[X^{(N)}(u) > N^{\varepsilon} Y^{(N)}(u)\right] < N^{-D}.$$

We denote this by $X \prec Y$ or $X = \mathcal{O}_\prec(Y)$.

For two deterministic quantities $X, Y \in \mathbb{R}$ dependent on N, we write $X \ll Y$ if there exists $\varepsilon, N_0 > 0$ such that $|X| \leq N^{-\varepsilon}|Y|$ for all $N \geq N_0$.
The following proposition encompasses the main properties of stochastic domination.

Proposition 2.3. *(Proposition 6.5 in [3])*
(1) $\mathcal{X} \prec \mathcal{Y}$ and $\mathcal{Y} \prec \mathcal{Z}$ imply $\mathcal{X} \prec \mathcal{Z}$;
(2) $\mathcal{X}_1 \prec \mathcal{Y}_1$ and $\mathcal{X}_2 \prec \mathcal{Y}_2$ imply $\mathcal{X}_1 + \mathcal{X}_2 \prec \mathcal{Y}_1 + \mathcal{Y}_2$ and $\mathcal{X}_1\mathcal{X}_2 \prec \mathcal{Y}_1\mathcal{Y}_2$;
(3) $\mathcal{X} \prec \mathcal{Y}$, $\mathbb{E}[\mathcal{Y}] \geq N^{-c}$ and $|\mathcal{X}| \leq N^c$ almost surely for some positive c imply $\mathbb{E}[\mathcal{X}] \prec \mathbb{E}[\mathcal{Y}]$.

We use C and c to denote constants the value of which may change from line to line.

2.3 Linear eigenvalue statistics

Let g be a complex-valued $C_c^2(\mathbb{R})$ function. Fix an energy E_0 in the bulk of the self-consistent spectrum and let $N^{-1} \ll \eta_0 \ll 1$.
We focus on studying scaled test functions given by

$$f(x) := g\left(\frac{x - E_0}{\eta_0}\right) \tag{2.3}$$

The goal is to show that

$$\frac{\operatorname{Tr} f(H) - \mathbb{E}[\operatorname{Tr} f(H)]}{\sqrt{V(f)}} \to \mathcal{N}(0, 1) \tag{2.4}$$

in distribution as $N \to \infty$, for some functional V which expresses the variance.
This can be accomplished by using the characteristic function method.

3 Local Law and Preliminaries

For a symmetric matrix H and a spectral parameter $z \in \mathbb{C}$, let $G(z)$ denote the resolvent of H.

$$G(z) = (H - z)^{-1}. \tag{3.1}$$

Let $\mathbf{m}(z) = (\mathbf{m}_j(z))_{j=1}^N$ be the solution to the quadratic vector Dyson equation,

$$\frac{-1}{\mathbf{m}_j(z)} = z + \sum_{k=1}^N S_{jk}\mathbf{m}_k(z), \quad j \in \{1, \dots, N\}, \quad \operatorname{Im} z > 0. \tag{3.2}$$

By Theorem 6.1.4 in [4], the solution $\mathbf{m}(z)$ is unique under the condition $\operatorname{Im}\mathbf{m}(z) > 0$ and can be extended into the lower complex half-plane by setting $\mathbf{m}(\bar{z}) := \overline{\mathbf{m}(z)}, \operatorname{Im} z > 0$.
Define $m(z)$ to be the average of the coordinates of $\mathbf{m}(z)$,

$$m(z) = \frac{1}{N}\sum_{j=1}^N \mathbf{m}_j(z). \tag{3.3}$$

Lemma 3.1. *The solution $\mathbf{m}(z)$ of (3.2) satisfies the following properties*
(1) (Theorem 6.1.4 in [4]) For every $\jmath \in \{1, \dots, N\}$ there exists a generating probability measure $\nu_j(\mathrm{d}x)$ such that

$$\mathbf{m}_j(z) = \int_{\mathbb{R}} \frac{\nu_j(\mathrm{d}x)}{x - z}. \tag{3.4}$$

(2) (Theorem 7.2.2 in [4]) Let $\rho(z) := \operatorname{Im} m(z)/\pi$ be the harmonic extension of the average of ν_j.
If the matrix of variances S satisfies the condition (A), then for all $z \in \mathbb{C}\backslash\mathbb{R}$ and all $j \in \{1, \dots, N\}$, the solution admits the following bounds

$$|\mathbf{m}_j(z)| \le \frac{c}{\rho(z) + \operatorname{dist}(z, \operatorname{supp}\rho)}, \quad \frac{1}{|\mathbf{m}_j(z)|} \le C(1 + |z|). \tag{3.5}$$

Let $\tau > 0$. We define the spectral domain

$$\mathcal{D} := \{z \in \mathbb{C} : N^{-1+\tau} \le \operatorname{Im} z \le \tau^{-1}, |\operatorname{Re} z| \le \tau^{-1}\}, \tag{3.6}$$

and two control parameters (as in [1]) for $z = E + i\eta \in \mathbb{C}\backslash\mathbb{R}$,

$$\Psi(z) := \sqrt{\frac{|\operatorname{Im} m(z)|}{N|\eta|}} + \frac{1}{N|\eta|}, \quad \Theta(z) := \frac{1}{N|\eta|}. \tag{3.7}$$

We give the following definition to the bulk of the self-consistent spectrum.
Let $\mathcal{I}$ be the set on which the generating measure ν_j defined in Lemma 3.1 is positive. Assumption (B) and Theorem 6.1.13 of [4] guarantee that the set $\mathcal{I}$ does not depend on the index j and consist of a finite union of open intervals $(\alpha^{(j)}, \beta^{(j)})$. Then for $\kappa > 0$ we define the bulk by

$$\mathcal{I}_\kappa := \bigcup_j [\alpha^{(j)} + \kappa, \beta^{(j)} - \kappa], \tag{3.8}$$

and the bulk domain

$$\mathcal{D}_\kappa := \{z \in \mathcal{D} : \operatorname{Re} z \in \mathcal{I}_\kappa\}. \tag{3.9}$$

In particular, for all $z \in \mathcal{I}_\kappa$

$$\rho(z) \ge C(\kappa), \tag{3.10}$$

for some positive constant $C(\kappa)$ dependent of κ.

Theorem 3.2. *Let w, x, y be deterministic vectors in $\mathbb{C}^N$ satisfying $\|w\|_\infty = 1$ and $\|x\|_2 = \|y\|_2 = 1$. Then the following estimates hold uniformly in $z \in \mathcal{D}$:*

$$|G_{ij}(z) - \delta_{ij}\mathbf{m}_j(z)| \prec \Psi(z), \tag{3.11}$$

$$\left| \frac{1}{N} \sum_{j=1}^{N} w_j \left(G_{jj}(z) - \mathbf{m}_j(z) \right) \right| \prec \Theta(z), \tag{3.12}$$

$$\left| \langle x, G(z) y \rangle - \sum_{j=1}^{N} \mathbf{m}_j(z) \bar{x}_j y_j \right| \prec \Psi(z). \tag{3.13}$$

We introduce the stability operator defied by the matrix

$$\delta_{ij} - S_{ij} \mathbf{m}_j(z) \mathbf{m}_j(\zeta), \quad i, j \in \{1, \ldots, N\}, \quad z, \zeta \in \mathbb{C} \backslash \mathbb{R}, \tag{3.14}$$

and denoted by $1 - S\mathbf{m}(z)\mathbf{m}(\zeta)$ for convenience of notation.
The stability analysis relies on the corresponding saturated self-energy operator F is defined by

$$F_{ij}(z, \zeta) := |\mathbf{m}_i(z)| S_{ij} |\mathbf{m}_j(\zeta)|. \tag{3.15}$$

Proposition 3.3. *(Proposition 4.5 in [2], analogous to Proposition 7.2.9 in [4]) Let $F(z, \zeta)$ be an operator defined by* (3.15)*. For any $z, \zeta \in \mathbb{C} \backslash \mathbb{R}$ it is a symmetric matrix with positive entries, hence by Perron–Frobenius theorem its principal eigenvalue is positive and simple and the corresponding ℓ^2-normalized eigenvector $v(z, \zeta)$ has strictly positive entries.*
We have the following upper bound for the norm:

$$\|F(z, \zeta)\|_{\ell^2 \to \ell^2} \le 1 - \frac{1}{2} \left(|\operatorname{Im} z| \frac{\langle v(z, z), |\mathbf{m}(z)| \rangle}{\langle v(z, z), \frac{|\operatorname{Im} \mathbf{m}(z)|}{|\mathbf{m}(z)|} \rangle} + |\operatorname{Im} \zeta| \frac{\langle v(\zeta, \zeta), |\mathbf{m}(\zeta)| \rangle}{\langle v(\zeta, \zeta), \frac{|\operatorname{Im} \mathbf{m}(\zeta)|}{|\mathbf{m}(\zeta)|} \rangle} \right). \tag{3.16}$$

Furthermore, for some positive constant c

$$\|F(z, \zeta)\|_{\ell^2 \to \ell^2} \le 1 - c \left(|\operatorname{Im} z| + |\operatorname{Im} \zeta| \right), \tag{3.17}$$

and the entries of $v(z, \zeta)$ are comparable in size

$$\frac{c}{\sqrt{N}} \le v_j(z, \zeta) \le \frac{C}{\sqrt{N}}, \quad j \in \{1, \ldots, N\}. \tag{3.18}$$

By combining this with (3.5), it follows immediately that the stability operator $1 - S\mathbf{m}(z)\mathbf{m}(\zeta)$ is invertible.

Lemma 3.4. *Cumulant expansion. (Lemma 4.2 in [1]) Let h be a real-valued random variable with finite moments, let f be a $C^\infty(\mathbb{R})$ function. Then for any $l \in \mathbb{N}$ the following expansion holds,*

$$\mathbb{E}\left[h \cdot f(h) \right] = \sum_{j=0}^{l} \frac{1}{j!} c^{(j+1)}(h) \, \mathbb{E}\left[\frac{\mathrm{d}^j}{\mathrm{d}h^j} f(h) \right] + R_{l+1}, \tag{3.19}$$

where $c^{(j)}$ is the j-th cumulant of h defined by

$$c^{(j)}(h) = (-i)^j \, \frac{\mathrm{d}^j}{\mathrm{d}t^j} \left(\log \mathbb{E}\left[e^{ith} \right] \right) \Big|_{t=0},$$

and the remainder term R_{l+1} satisfies

$$|R_{l+1}| \le C_l \, \mathbb{E}\left[|h|^{l+2} \right] \sup_{|x| \le M} |f^{(l+1)}(x)| + C_l \, \mathbb{E}\left[|h|^{l+2} \cdot \mathbb{1}_{|h| > M} \right] \left\| f^{(l+1)}(x) \right\|_\infty, \tag{3.20}$$

for any $M > 0$.

4 Result

Theorem 4.1. *Let the matrix of variances S satisfy assumptions* (A) *and* (B). *Assume that the stability operator defined in* (3.14) *admits the following bound*

$$\left\|(1 - S\mathbf{m}(z)\mathbf{m}(\zeta))^{-1}\right\|_{\ell^2\to\ell^2} \le \frac{C}{|\operatorname{Im} z| + |\operatorname{Im}\zeta| + |\zeta - \bar{z}|}, \tag{4.1}$$

for $z, \zeta \in \mathcal{D}_\kappa$ with $\operatorname{Im} z \operatorname{Im}\zeta < 0$ and $\operatorname{Re} z, \operatorname{Re}\zeta$ in the same connected component of $\mathcal{I}_\kappa$ defined in (3.8). *Assume additionally that the following estimate holds*

$$\sum_{i,j,k=1}^{N} \left((1 - \mathbf{m}^2(z)S)^{-1}\right)_{ij} \mathbf{m}_j(z)\Pi_{jk}(z)T_{kj}(z,\zeta) = \mathcal{T}(z,\zeta) + \mathcal{O}_\prec(\Theta(z) + \Theta(\zeta))\eta^{-1}). \tag{4.2}$$

where the main term $\mathcal{T}(z,\zeta)$ is a deterministic quantity and the projector Π is defined as follows

$$\Pi(z) := |\mathbf{m}(z)|^{-1} w(z) w(z)^* |\mathbf{m}(z)|, \quad w(z) := \frac{\operatorname{Im}\mathbf{m}(z)}{|\mathbf{m}(z)|}\left\|\frac{\operatorname{Im}\mathbf{m}(z)}{|\mathbf{m}(z)|}\right\|_2^{-1}. \tag{4.3}$$

For f defined in (2.3) *with $N^{-1} \ll \eta_0 \ll 1$ and $E_0 \in \mathcal{I}_\kappa$ as in* (3.8), *define*

$$V(f) := \frac{1}{\pi^2}\int_{\Omega_\alpha}\int_{\Omega_\alpha} \frac{\partial \tilde{f}(\zeta)}{\partial\bar{\zeta}}\frac{\partial\tilde{f}(z)}{\partial\bar{z}}\mathcal{K}(z,\zeta)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta, \tag{4.4}$$

where

$$\begin{aligned}\mathcal{K}(z,\zeta) := \sum_{i,j=1}^{N} \left((1 - \mathbf{m}(z)^2 S)^{-1}\right)_{ij} \mathbf{m}_j(z)\Bigg(2\frac{\partial}{\partial\zeta}\left(\sum_{k=1}^{N}\big(1 - \Pi(z)\big)_{jk}\left((1 - S\mathbf{m}(z)\mathbf{m}(\zeta))^{-1}(S\mathbf{m}(z)\mathbf{m}(\zeta))^2\right)_{kj}\right)\\ + 2\frac{\partial\mathcal{T}(z,\zeta)}{\partial\zeta} + S_{jj}\mathbf{m}_j'(\zeta)\mathbf{m}_j(z) + \sum_{k=1}^{N} c^{(4)}(H_{jk})\mathbf{m}_j(z)\mathbf{m}_k(z)\frac{\partial\left(\mathbf{m}_j(\zeta)\mathbf{m}_k(\zeta)\right)}{\partial\zeta}\Bigg).\end{aligned} \tag{4.5}$$

Then, if there exist positive constants c and C such that $c \le V(f) \le C$,

$$\frac{\operatorname{Tr} f(H) - \mathbb{E}\left[\operatorname{Tr} f(H)\right]}{\sqrt{V(f)}} \to \mathcal{N}(0,1). \tag{4.6}$$

5 Proof

Following [1] and [2] we employ the characteristic function method.
We begin by introducing the quasi-analytic extension of the scaled test function f as defined in (2.3)

$$\tilde{f}(x + iy) = \chi(y)\left(f(x) + iyf'(x)\right), \tag{5.1}$$

where $\chi : \mathbb{R} \to [-1,1]$ is an even $\chi \in C_c^\infty(\mathbb{R})$ function supported on $[-1,1]$ satisfying $\chi(y) = 1$ for $|y| < \frac{1}{2}$.

Using the Helffer–Sjöstrand representation we can express the linear eigenvalue statistics in terms of the resolvent of H. This is a crucial step, because it "transfers the randomness" from the function to the resolvent and allows us to proceed by using the local laws of Theorem 3.2 for $G(z)$.

$$\{1 - \mathbb{E}\}\left[\operatorname{Tr} f(H)\right] = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{f}}{\partial\bar{z}}\{1 - \mathbb{E}\}\left[\operatorname{Tr} G(z)\right]\mathrm{d}\bar{z}\mathrm{d}z. \tag{5.2}$$

The characteristic function of linear eigenvalue statistics then admits the following form

$$\phi(\lambda) := \mathbb{E}\left[e(\lambda)\right], \quad e(\lambda) := \exp\left\{i\lambda\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\tilde{f}}{\partial\bar{z}}\{1 - \mathbb{E}\}\left[\operatorname{Tr} G(z)\right]\mathrm{d}\bar{z}\mathrm{d}z\right\}, \quad \lambda \in \mathbb{R}. \tag{5.3}$$

Differentiating with respect to λ we get

$$\phi'(\lambda) = \mathbb{E}\left[e(\lambda) \frac{i}{\pi} \int\limits_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \{1 - \mathbb{E}\} [\operatorname{Tr} G(z)] \, \mathrm{d}\bar{z}\mathrm{d}z \right]. \tag{5.4}$$

Computing the partial derivatives of (5.1) with respect to x and y we get

$$\frac{\partial \tilde{f}}{\partial \bar{z}} = \frac{1}{2}\Big(-y\chi'(y)f'(x) + i\big(y\chi(y)f''(x) + \chi'(y)f(x)\big) \Big). \tag{5.5}$$

Observe that the fact that χ is even implies

$$\frac{\partial \tilde{f}}{\partial \bar{z}}(x - iy) = \overline{\frac{\partial \tilde{f}}{\partial \bar{z}}(x + iy)}. \tag{5.6}$$

It follows that the imaginary part of integrand in (5.2) is odd with respect to $\operatorname{Im} z$, so the integral is real and $|e(\lambda)| = 1$.

5.1 Non-contribution of the ultra-local scales

We eliminate a small vicinity of the real line from the integral in (5.2) using the argument from the proof of Proposition 4.1 in [5].
Fix $\alpha \in (0,1)$, such that $N^{-\alpha}\eta_0 \gg N^{-1}$, where η_0 is in (2.3), and define

$$\Omega_\alpha := \{z \in \mathbb{C} : |\operatorname{Im} z| > N^{-\alpha}\eta_0\}. \tag{5.7}$$

For all $x + iy \in \Omega_\alpha^c$ we have $\chi(y) = 1, \chi'(y) = 0$, hence by (5.5) the derivative of the quasi-analytic extension takes the form

$$\frac{\partial \tilde{f}}{\partial \bar{z}} = \frac{iy}{2} f''(x), \quad x + iy \in \Omega_\alpha^c. \tag{5.8}$$

Note that $G(\bar{z}) = \overline{G(z)}$ hence we can write

$$\left| \int\limits_{\Omega_\alpha^C} \frac{\partial \tilde{f}}{\partial \bar{z}} \{1 - \mathbb{E}\} [\operatorname{Tr} G(z)] \, \mathrm{d}\bar{z}\mathrm{d}z \right| \quad \leq \quad \int\limits_{\mathbb{R}} |f''(x)| \left| \int\limits_0^{N^{-\alpha}\eta_0} y \operatorname{Im}\left(\{1 - \mathbb{E}\} [\operatorname{Tr} G(x+iy)]\right) \mathrm{d}y \right| \mathrm{d}x \tag{5.9}$$

We estimate the integral over y by considering two regimes:
For $\operatorname{Im} z = y \geq y_0 := \sqrt{\eta_0 \cdot N^{-1-\alpha}}$ we apply the local law (3.12) with $w_j = 1$ to get

$$\left| \int\limits_{y_0}^{N^{-\alpha}\eta_0} y \operatorname{Im}\left(\{1 - \mathbb{E}\} [\operatorname{Tr} G(x+iy)]\right) \mathrm{d}y \right| \prec \int\limits_{y_0}^{N^{-\alpha}\eta_0} 2y \frac{N}{Ny} \mathrm{d}y = 2\left(N^{-\alpha}\eta_0 - y_0\right) = \mathcal{O}_\prec(N^{-\alpha}\eta_0), \tag{5.10}$$

because $N^{-\alpha}\eta_0$ implies $y_0 \ll N^{-\alpha}\eta_0$.
For $\operatorname{Im} z = y \in (0, y_0)$ we use the fact that the map $y \mapsto y \operatorname{Im} \operatorname{Tr} G(x + iy)$ is non-decreasing, hence

$$\left| \int\limits_0^{y_0} y \operatorname{Im}\left(\{1 - \mathbb{E}\} [\operatorname{Tr} G(z)]\right) \mathrm{d}y \right| \leq y_0^2 \left(|\operatorname{Im} \operatorname{Tr} G(x+iy_0)| + |\mathbb{E}[\operatorname{Im} \operatorname{Tr} G(x+iy_0)]| \right) \prec 4Ny_0^2, \tag{5.11}$$

where the last estimate follows again from the local law (3.12) and the bounds in (3.5). By definition of y_0 we have $Ny_0^2 = N^{-\alpha}\eta_0$.
Finally, $\|f''\|_1 = \|g''\|_1 / \eta_0$, which implies

$$\int\limits_{\Omega_\alpha^c} \frac{\partial \tilde{f}}{\partial \bar{z}} \{1 - \mathbb{E}\} [\operatorname{Tr} G(z)] \, \mathrm{d}\bar{z}\mathrm{d}z \prec \int\limits_{\mathbb{R}} |f''(x)| \mathrm{d}x \cdot \mathcal{O}_\prec(N^{-\alpha}\eta_0 + Ny_0^2) = \mathcal{O}_\prec(N^{-\alpha}). \tag{5.12}$$

Furthermore, it follows from $|e(\lambda)| = 1$ that

$$\mathbb{E}\left[e(\lambda)\frac{i}{\pi}\int\limits_{\Omega_\alpha^c}\frac{\partial\tilde{f}}{\partial\bar{z}}\{1-\mathbb{E}\}[\operatorname{Tr}G(z)]\,\mathrm{d}\bar{z}\mathrm{d}z\right] = \mathcal{O}_\prec(N^{-\alpha}) \tag{5.13}$$

Hence, we can write

$$\phi'(\lambda) = \frac{i}{\pi}\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{z}}\,\mathbb{E}\left[\{1-\mathbb{E}\}[\operatorname{Tr}G(z)]\,e(\lambda)\right]\mathrm{d}\bar{z}\mathrm{d}z + \mathcal{O}_\prec(N^{-\alpha}). \tag{5.14}$$

Define

$$\tilde{e}(\lambda) := \exp\left\{i\lambda\frac{1}{\pi}\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{z}}\{1-\mathbb{E}\}[\operatorname{Tr}G(x+iy)]\,\mathrm{d}\bar{z}\mathrm{d}z\right\}, \tag{5.15}$$

then by (5.12) we have

$$e(\lambda) - \tilde{e}(\lambda) = \mathcal{O}_\prec(|\lambda|N^{-\alpha}), \tag{5.16}$$

which by local law (3.12) yields

$$\mathbb{E}\left[\{1-\mathbb{E}\}[\operatorname{Tr}G(z)]\cdot(e(\lambda)-\tilde{e}(\lambda))\right] = \mathcal{O}(|\lambda|N^{-\alpha}\eta^{-1}). \tag{5.17}$$

Further estimates will require the following technical lemma.

Lemma 5.1. *(Lemma 5.4 in [2]) Let $K(z)$ be a holomorphic function on $\mathbb{C}\backslash\mathbb{R}$ satisfying*

$$|K(z)| \le \frac{C}{|\operatorname{Im} z|^s},$$

for some $1 \le s \le 2$. Then there exists a constant $C' > 0$ such that

$$\left|\int\limits_{\Omega_\alpha} iy\chi(y)f''(x)K(x+iy)\mathrm{d}x\mathrm{d}y\right| \le CC'\log N(1+\|f''\|_1)^{s-1}.$$

Using the estimate (5.17) and applying Lemma 5.1 for $K(z) = \mathbb{E}\left[\{1-\mathbb{E}\}[\operatorname{Tr}G(z)]\cdot(e(\lambda)-\tilde{e}(\lambda))\right]$ with $s = 1$ we get from (5.14)

$$\phi'(\lambda) = \frac{i}{\pi}\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{z}}\,\mathbb{E}\left[\tilde{e}(\lambda)\{1-\mathbb{E}\}[\operatorname{Tr}G(z)]\right]\mathrm{d}\bar{z}\mathrm{d}z + \mathcal{O}_\prec(|\lambda|N^{-\alpha}), \tag{5.18}$$

note that $\log N$ factor in the error term of Lemma 5.1 is absorbed by stochastic domination.
We aim to show that $\phi'(\lambda)$ is proportional to $-\lambda\,\mathbb{E}[\tilde{e}(\lambda)]$ up to a negligible error. To achieve this we need to estimate $\mathbb{E}[\tilde{e}(\lambda)\{1-\mathbb{E}\}[\operatorname{Tr}G(z)]]$ by using cumulant expansion formula (3.4) following the approach laid out in the proof of Proposition 4.1 of [1] and Lemma 5.7 of [2].

5.2 Cumulant expansion

The goal of this subsection is to prove the following statement.

Proposition 5.2. *For any $z \in \mathcal{D}$ defined in* (3.6) *and all indices $j \in \{1, \ldots, N\}$ we have*

$$\begin{aligned}\frac{-1}{\mathbf{m}_j(z)}\mathbb{E}\left[\{1-\mathbb{E}\}\left[G_{jj}(z)\right]\tilde{e}(\lambda)\right] = &-\mathbf{m}_j(z)\sum_{k=1}^{N} S_{jk}\,\mathbb{E}\left[\{1-\mathbb{E}\}\left[G_{kk}(z)\right]\tilde{e}(\lambda)\right] \\ &-\mathbb{E}\left[\{1-\mathbb{E}\}\left[T_{jj}(z,z)\right]\tilde{e}(\lambda)\right] \\ &-\frac{2i\lambda}{\pi}\,\mathbb{E}\left[\tilde{e}(\lambda)\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\frac{\partial T_{jj}(z,\zeta)}{\partial\zeta}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right] \\ &-\frac{i\lambda}{\pi}S_{jj}\,\mathbb{E}\left[\tilde{e}(\lambda)\right]\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\mathbf{m}_j'(\zeta)\mathbf{m}_j(z)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta \\ &-\frac{i\lambda}{\pi}\,\mathbb{E}\left[\tilde{e}(\lambda)\right]\sum_{k=1}^{N}c^{(4)}(H_{jk})\mathbf{m}_j(z)\mathbf{m}_k(z)\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\frac{\partial(\mathbf{m}_j(\zeta)\mathbf{m}_k(\zeta))}{\partial\zeta}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta \\ &+\mathcal{O}_\prec(\Psi(z)\Theta(z)+N^{-1}\Psi(z)(1+|\lambda|^2)\eta_0^{-1/2}+N^{-3/2}(1+|\lambda|)^4),\end{aligned} \tag{5.19}$$

where η_0 is from (2.3)*, and for $a, b \in \{1, \ldots, N\}$, $z, \zeta \in \mathbb{C}\backslash\mathbb{R}$ we define*

$$T_{ab}(z,\zeta) := \sum_{j\neq b} S_{aj}G_{jb}(z)G_{jb}(\zeta). \tag{5.20}$$

Note that after multiplying both sides of (5.19) by $-\mathbf{m}_j(z)$ and carrying the first term of the right-hand side to the left, we obtain an expression for $\mathbb{E}\left[\{1-\mathbb{E}\}\left[G_{jj}(z)\right]\tilde{e}(\lambda)\right]$ by inverting the operator $1-\mathbf{m}^2(z)S$. In the bulk the inverse of this operator, $\left(1-\mathbf{m}^2(z)S\right)^{-1}$, is bounded by a constant as shown in Lemma 7.3.2 of [4].

Proof of Proposition 5.2.
We begin by stating the resolvent identity,

$$zG_{ij}(z) = \sum_{k=1}^{N} H_{ik}G_{kj}(z) - \delta_{ij}, \quad i,j \in \{1,\ldots,N\}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{5.21}$$

which follows immediately from the definition of resolvent (3.1).
Observe that

$$\mathbb{E}\left[\mathcal{X}\{1-\mathbb{E}\}[\mathcal{Y}]\right] = \mathbb{E}\left[\{1-\mathbb{E}\}[\mathcal{X}]\mathcal{Y}\right], \tag{5.22}$$

for any random variables $\mathcal{X}, \mathcal{Y}$ with $\mathbb{E}[\mathcal{X}], \mathbb{E}[\mathcal{Y}], \mathbb{E}[\mathcal{X}\mathcal{Y}] < \infty$.
Applying identities (5.21) and (5.22) we get

$$z\,\mathbb{E}\left[\{1-\mathbb{E}\}\left[G_{jj}(z)\right]\tilde{e}(\lambda)\right] = \sum_{k=1}^{N}\mathbb{E}\left[\{1-\mathbb{E}\}\left[H_{jk}G_{kj}(z)\right]\tilde{e}(\lambda)\right] = \sum_{k=1}^{N}\mathbb{E}\left[H_{jk}\cdot G_{kj}(z)\{1-\mathbb{E}\}\left[\tilde{e}(\lambda)\right]\right], \tag{5.23}$$

which allows us to use the cumulant expansion Lemma (3.4) with $l = 3$.

$$z\,\mathbb{E}\left[\{1-\mathbb{E}\}\left[G_{jj}(z)\right]\tilde{e}(\lambda)\right] = \sum_{k=1}^{N}\left(\sum_{s=0}^{3}\frac{1}{s!}c^{(s+1)}(H_{jk})\,\mathbb{E}\left[\frac{\partial^s}{\partial H_{jk}^s}\left(G_{kj}(z)\{1-\mathbb{E}\}\left[\tilde{e}(\lambda)\right]\right)\right] + R_{jk}^{(4)}\right), \tag{5.24}$$

We denote the terms corresponding to each cumulant order summed over k by I_s for $s \in \{1,2,3\}$, omitting the dependence on j for convenience. The $s = 0$ term vanishes since $\mathbb{E}[H_{jk}] = 0$.

First we focus on I_1. Recall that $c^{(2)}(H_{jk}) = \mathbb{E}\left[|H_{jk}|^2\right] = S_{jk}$, hence

$$I_1 = \sum_{k=1}^{N} S_{jk}\,\mathbb{E}\left[\{1-\mathbb{E}\}\left[\frac{\partial G_{kj}(z)}{\partial H_{jk}}\right]\tilde{e}(\lambda) + G_{kj}(z)\frac{\partial\tilde{e}(\lambda)}{\partial H_{jk}}\right]. \tag{5.25}$$

Partial derivatives of the resolvent entries with respect to matrix elements have the following expression.

$$\frac{\partial G_{ab}(z)}{\partial H_{jk}} = -\frac{G_{aj}(z)G_{kb}(z) + G_{bj}(z)G_{ka}(z)}{1+\delta_{jk}}. \tag{5.26}$$

By separating the $k = j$ term, the local law (3.11) and (3.5) then yields:

$$\begin{aligned}\sum_{k=1}^{N} S_{jk}\,\mathbb{E}\left[\{1-\mathbb{E}\}\left[\frac{\partial G_{kj}(z)}{\partial H_{jk}}\right]\tilde{e}(\lambda)\right] = -\mathbf{m}_j(z)\sum_{k\neq j}\mathbb{E}\left[\{1-\mathbb{E}\}\left[G_{kk}(z)\right]\tilde{e}(\lambda)\right] \\ -\mathbb{E}\left[\{1-\mathbb{E}\}\left[T_{jj}(z,z)\right]\right] + \mathcal{O}_{\prec}(N^{-1}).\end{aligned} \tag{5.27}$$

We compute the partial derivatives of $\tilde{e}$ defined in (5.15):

$$\frac{\partial \tilde{e}(\lambda)}{\partial H_{jk}} = -\frac{2i\lambda}{\pi}\tilde{e}(\lambda)\int_{\Omega_\alpha}\frac{\partial \tilde{f}}{\partial \bar{\zeta}}\frac{\partial G_{kj}(\zeta)}{\partial \zeta}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta. \tag{5.28}$$

We require the following lemma, which is an immediate consequence of the Cauchy's integral formula for the derivatives of an analytic function. It will be applied to obtain local law analogues for the derivatives of functions.

Lemma 5.3. *(Lemma 5.5 in [2]) Let $K(z)$ be a holomorphic function on $\mathbb{C}\backslash\mathbb{R}$. Then for any $z \in \mathbb{C}\backslash\mathbb{R}$ and any $p \in \mathbb{N}$,*

$$\left|\frac{\partial^p K}{\partial z^p}(z)\right| \leq C_k|\operatorname{Im} z|^{-p}\sup_{|\zeta - z|\leq|\operatorname{Im} z|/2}|K(\zeta)|. \tag{5.29}$$

Observe that $K(z) := G_{jj}(z) - \mathbf{m}_j(z)$ is analytic and bounded by $\mathcal{O}_{\prec}(\Psi(z))$, hence by Lemma 5.3 with $p = 1$ we have the following bound

$$\frac{\partial G_{jj}(z)}{\partial z} - \mathbf{m}_j'(z) = \mathcal{O}_{\prec}\left(\frac{\Psi(z)}{|\operatorname{Im} z|}\right). \tag{5.30}$$

It follows by from Lemma 5.1 applied to the integral in (5.28) with $k = j$, $K(z) = \frac{\partial G_{jj}(z)}{\partial z} - \mathbf{m}_j'(z)$ and $s = 3/2$, that

$$\frac{\partial \tilde{e}(\lambda)}{\partial H_{jj}} = -\frac{2i\lambda}{\pi}\tilde{e}(\lambda)\int_{\Omega_\alpha}\frac{\partial \tilde{f}}{\partial \bar{\zeta}}\mathbf{m}_j'(\zeta)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta + \mathcal{O}_{\prec}(|\lambda|\eta_0^{-1/2}). \tag{5.31}$$

Summing the remaining terms with the derivative of $\tilde{e}$ over $k \neq j$ we have from (5.20)

$$\sum_{k\neq j} S_{jk}\,\mathbb{E}\left[G_{kj}(z)\frac{\partial \tilde{e}(\lambda)}{\partial H_{jk}}\right] = -\frac{2i\lambda}{\pi}\,\mathbb{E}\left[\tilde{e}(\lambda)\int_{\Omega_\alpha}\frac{\partial \tilde{f}}{\partial \bar{\zeta}}\frac{\partial T_{jj}(z,\zeta)}{\partial \zeta}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right]. \tag{5.32}$$

Plugging (5.27), (5.31) and (5.32) into (5.25) and applying local laws (3.11), (3.12) we get

$$\begin{aligned}I_1 = &\left(z + \frac{1}{\mathbf{m}_j(z)}\right)\mathbb{E}\left[\{1-\mathbb{E}\}\left[G_{jj}(z)\right]\tilde{e}(\lambda)\right] - \mathbf{m}_j(z)\sum_{k=1}^{N} S_{jk}\,\mathbb{E}\left[\{1-\mathbb{E}\}\left[G_{kk}(z)\right]\tilde{e}(\lambda)\right] \\ &- \mathbb{E}\left[\{1-\mathbb{E}\}\left[T_{jj}(z,z)\right]\tilde{e}(\lambda)\right] - \frac{2i\lambda}{\pi}\,\mathbb{E}\left[\tilde{e}(\lambda)\int_{\Omega_\alpha}\frac{\partial \tilde{f}}{\partial \bar{\zeta}}\frac{\partial T_{jj}(z,\zeta)}{\partial \zeta}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right] \\ &- \frac{i\lambda}{\pi}S_{jj}\,\mathbb{E}\left[\tilde{e}(\lambda)\int_{\Omega_\alpha}\frac{\partial \tilde{f}}{\partial \bar{\zeta}}\mathbf{m}_j'(\zeta)\mathbf{m}_j(z)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right] \\ &+ \mathcal{O}_{\prec}(\Psi(z)\Theta(z) + N^{-1}\Psi(z)|\lambda|\eta_0^{-1/2}),\end{aligned} \tag{5.33}$$

where $T_{ab}(z,\zeta)$ is defined in (5.20).
Note that the factor z in the first term cancels out with the left-hand side of (5.23) and $1/\mathbf{m}_j(z)$ contributes to the left-hand side of (5.19).

Working with I_2 we apply similar reasoning. Note that $c^{(3)}(H_{jk}) = \mathbb{E}\left[|H_{jk}|^3\right]$ and hence by moment condition (2.2) the third cumulants are of order $N^{-3/2}$.

$$I_2 = \sum_{k=1}^{N} c^{(3)}(H_{jk})\,\mathbb{E}\left[\{1-\mathbb{E}\}\left[\frac{\partial^2 G_{kj}(z)}{\partial H_{jk}^2}\right]\tilde{e}(\lambda) + 2\frac{\partial G_{kj}(z)}{\partial H_{jk}}\frac{\partial \tilde{e}(\lambda)}{\partial H_{jk}} + G_{kj}(z)\frac{\partial^2 \tilde{e}(\lambda)}{\partial H_{jk}^2}\right]. \tag{5.34}$$

We start by computing the second derivatives of $\tilde{e}$ defined in (5.15) :

$$\begin{aligned}\frac{\partial^2 \tilde{e}(\lambda)}{\partial H_{jk}^2} &= \tilde{e}(\lambda)\left(\frac{2i\lambda}{\pi}\int_{\Omega_\alpha}\frac{\partial \tilde{f}}{\partial \bar{\zeta}}\frac{\partial G_{kj}(\zeta)}{\partial \zeta}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right)^2 \\ &\quad + \tilde{e}(\lambda)\frac{2i\lambda}{\pi}\int_{\Omega_\alpha}\frac{\partial \tilde{f}}{\partial \bar{\zeta}}\frac{1}{(1+\delta_{jk})^2}\frac{\partial}{\partial \zeta}\left\{G_{jj}(\zeta)G_{kk}(\zeta) + G_{jk}(\zeta)G_{kj}(\zeta)\right\}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta.\end{aligned} \tag{5.35}$$

For all $k \neq j$, the local law (3.11), Lemmas 5.3 and 5.1 applied to (5.28) give the following bound.

$$\left|\frac{\partial \tilde{e}(\lambda)}{\partial H_{jk}}\right| = \mathcal{O}_\prec(N^{-1/2}(1+|\lambda|)\eta_0^{-1/2}). \tag{5.36}$$

This implies that for $k \neq j$ the first term in (5.35) is negligible, applying the local law (3.11) and Lemma 5.3 we get

$$\frac{\partial^2 \tilde{e}(\lambda)}{\partial H_{jk}^2} = \tilde{e}(\lambda)\frac{2i\lambda}{\pi}\int_{\Omega_\alpha}\frac{\partial \tilde{f}}{\partial \bar{\zeta}}\frac{\partial(\mathbf{m}_j(\zeta)\mathbf{m}_k(\zeta))}{\partial \zeta}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta + \mathcal{O}_\prec(N^{-1/2}(1+|\lambda|)^2\eta_0^{-1/2}). \tag{5.37}$$

And for $k = j$ we use a universal bound found in Lemma 5.6 of [2]

$$\left|\frac{\partial^s \tilde{e}(\lambda)}{\partial H_{jk}^s}\right| = \mathcal{O}_\prec((1+|\lambda|)^s), \quad s \in \mathbb{N}, \quad j,k \in \{1,\dots,N\}. \tag{5.38}$$

Applying these bounds and the local laws from Theorem 3.2 to the (5.34) yields the following.

$$I_2 = \mathcal{O}_\prec(N^{-1}\Psi(z)(1+|\lambda|)^2\eta_0^{-1/2}). \tag{5.39}$$

Lastly, we compute the contribution of I_3. The fourth cumulant of H_{jk} can be expressed in terms of the fourth and the square of the second moments of H_{jk}, hence, by the moment condition (2.2), it will be of the order N^{-2}.

$$\begin{aligned}I_3 = \sum_{k=1}^{N} c^{(4)}(H_{jk})\,\mathbb{E}\Bigg[\{1-\mathbb{E}\}\left[\frac{\partial^3 G_{kj}(z)}{\partial H_{jk}^3}\right]\tilde{e}(\lambda) + 3\frac{\partial G_{kj}(z)}{\partial H_{jk}}\frac{\partial^2 \tilde{e}(\lambda)}{\partial H_{jk}^2} \\ + 3\frac{\partial^2 G_{kj}(z)}{\partial H_{jk}^2}\frac{\partial \tilde{e}(\lambda)}{\partial H_{jk}} + G_{kj}(z)\frac{\partial^3 \tilde{e}(\lambda)}{\partial H_{jk}^3}\Bigg].\end{aligned} \tag{5.40}$$

Since $c^{(4)}(H_{jj})$ is of order N^{-2} it is sufficient to handle the off-diagonal terms. For $k \neq j$ summing the last two terms and applying the local law (3.13) gives $\mathcal{O}_\prec(N^{-3/2}\Psi(z)(1+|\lambda|)^3)$. Summing the first term gives $\mathcal{O}_\prec(N^{-1}\Psi(z))$ by (3.11).
Hence by local law (3.11) and Lemma 5.3

$$I_3 = -3\sum_{k\neq j} c^{(4)}(H_{jk})\mathbf{m}_j(z)\mathbf{m}_k(z)\,\mathbb{E}\left[\frac{\partial^2 \tilde{e}(\lambda)}{\partial H_{jk}^2}\right] + \mathcal{O}_\prec(N^{-1}\Psi(z)(1+|\lambda|)^2) \tag{5.41}$$

And the computation is complete by plugging in the expression (5.37).

We finish the proof by using the bound (5.38) to get

$$R_{jk}^{(4)} = \mathcal{O}_\prec(N^{-5/2}(1+|\lambda|)^4). \tag{5.42}$$

□

5.3 Two-point function

To factor $\mathbb{E}[\tilde{e}(\lambda)]$ out of the second and third terms on the right-hand side of (5.19) we need to obtain an analogue of the local law for the two-point function $T_{jj}(z,\zeta)$.
Applying the method of Lemma 4.3 from [1], we find a self-consistent equation for $T_{ab}(z,\zeta)$.

Proposition 5.4. *For any $z,\zeta\in\mathcal{D}$ and any $a,b\in\{1,\dots,N\}$*

$$\Big((1-S\mathbf{m}(z)\mathbf{m}(\zeta))T(z,\zeta)\Big)_{ab}=\big((S\mathbf{m}(z)\mathbf{m}(\zeta))^2\big)_{ab}+\mathcal{O}_\prec(\Psi(z)^{3/2}\Psi(\zeta)+\Psi(z)\Psi(\zeta)^{3/2}). \tag{5.43}$$

Proof. It suffices to estimate all even moments $\mathbb{E}\left[|P_{ab}|^{2d}\right], d\in\mathbb{N}$ of the quantity

$$P_{ab}:=T_{ab}-\big(S\mathbf{m}(z)\mathbf{m}(\zeta)T\big)_{ab}-\big((S\mathbf{m}(z)\mathbf{m}(\zeta))^2\big)_{ab}. \tag{5.44}$$

We fix $j\in\{1,\dots,N\}$ and combine the vector Dyson equation (3.2) and the resolvent identity (5.21) to achieve the following equality

$$S_{aj}G_{jb}(z)G_{jb}(\zeta)=-\mathbf{m}_j(z)\sum_{\alpha=1}^{N}H_{j\alpha}G_{\alpha b}(z)S_{aj}G_{jb}(\zeta)-\mathbf{m}_j(z)\sum_{\alpha=1}^{N}S_{j\alpha}\mathbf{m}_\alpha(z)S_{aj}G_{jb}(z)G_{jb}(\zeta). \tag{5.45}$$

Summing over $j\neq b$ we get an expression for T_{ab} that we plug into the first term of (5.44). After this we apply the reasoning laid out in Section 5 of [1] to finish the proof. □

Observe that in order to express $T_{jj}(z,\zeta)$ from equation (5.43) one needs to invert the stability operator $1-S\mathbf{m}(z)\mathbf{m}(\zeta)$ and estimate the action of the inverse on the error term.

For $\zeta=z$, we will prove that the operator $1-S\mathbf{m}(z)^2$ is bounded in norm by a constant, so we can estimate the term on the right-hand side of (5.19) corresponding to $\{1-\mathbb{E}\}[T_{jj}(z,z)]$ by $\mathcal{O}_\prec(\Psi(z)^{5/2})$.

For $\zeta\neq z$, if the bound obtained from Proposition (3.3) is used, like it is done in [2], the relative error is of order $(\Psi(z)^{1/2}+\Psi(\zeta)^{1/2})\cdot(\operatorname{Im}z+\operatorname{Im}\zeta)^{-1}$, which is not negligible for η close to optimal N^{-1} scale.
The authors of [1] use a projection $1-\Pi$ onto the orthogonal complement of the principal eigenvector of S to improve the bound and estimate the value of $\operatorname{Tr}\Pi T$ separately. We aim to proceed in a similar fashion.

5.4 Stability operator bounds

The quantitative improvement of the bound on the norm of $\big(1-S\mathbf{m}(z)\mathbf{m}(\zeta)\big)^{-1}$ hinges on the fact that the self-saturated energy operator $F(z,\zeta)$ possesses a "large" spectral gap compared to the size of $1-\|F\|_{\ell^2\to\ell^2}$.
This statement is captured by the subsequent lemma, that follows directly from Lemma 7.4.2 of [4].

Lemma 5.5. *Let F be the self-saturated energy operator $F(z,\zeta)$ defined in* (3.15).
Then $\operatorname{Gap}F$*, i.e., the difference between the two largest eigenvalues of $|F|$, admits the following bound.*

$$\operatorname{Gap}F\geq\tilde{c}, \tag{5.46}$$

for a constant $\tilde{c}$ that depends on the constants in condition (A) *and inequality* (3.18).

Note that by (3.17) the quantity $1-\|F(z,\zeta)\|_{\ell^2\to\ell^2}$ is of order $|\operatorname{Im}z|+|\operatorname{Im}\zeta|\ll 1$, while $\operatorname{Gap}F(z,\zeta)$ is order 1.
This can be further connected to the norm of $\big(1-S\mathbf{m}(z)\mathbf{m}(\zeta)\big)^{-1}$ using the identity

$$1-S\mathbf{m}(z)\mathbf{m}(\zeta)=\mathrm{U}\,|\mathbf{m}(z)\mathbf{m}(\zeta)|^{-1/2}\,(\mathrm{U}^*-F(z,\zeta))\,|\mathbf{m}(z)\mathbf{m}(\zeta)|^{1/2},\quad \mathrm{U}:=\frac{\mathbf{m}(z)\mathbf{m}(\zeta)}{|\mathbf{m}(z)\mathbf{m}(\zeta)|}, \tag{5.47}$$

together with the following lemma.

Lemma 5.6. *(Lemma 7.4.7 in [4]) Let F be a hermitian matrix with $\|F\|_{\ell^2\to\ell^2}\leq 1$ and a principal ℓ^2-normalized eigenvector v, then for any unitary operator* U *we have*

$$\big\|(\mathrm{U}-F)^{-1}\big\|_{\ell^2\to\ell^2}\leq\frac{C}{\operatorname{Gap}F\cdot|1-\|F\|_{\ell^2\to\ell^2}\langle v,\mathrm{U}v\rangle|}. \tag{5.48}$$

Observe that by (3.18) the real part of the scalar product admits the following upper bound

$$\operatorname{Re}\langle v, \mathrm{U}v\rangle \leq 1 - \frac{c}{N}\sum_{j=1}^{N} \frac{\operatorname{Im}\mathbf{m}_j(z)\operatorname{Im}\mathbf{m}_j(\zeta)}{|\mathbf{m}_j(z)\mathbf{m}_j(\zeta)|}. \tag{5.49}$$

Let $\eta := \operatorname{Im} z$, $\eta' := \operatorname{Im}\zeta$. If $\eta\eta' > 0$, then $\operatorname{Im}\mathbf{m}(z)\operatorname{Im}\mathbf{m}(\zeta) > 0$ and we hence

$$1 - \|F(z,\zeta)\|_{\ell^2\to\ell^2}\operatorname{Re}\langle v, \mathrm{U}v\rangle \geq \|F(z,\zeta)\|_{\ell^2\to\ell^2}\frac{c}{N}\sum_{j=1}^{N}\frac{|\operatorname{Im}\mathbf{m}_j(z)||\operatorname{Im}\mathbf{m}_j(\zeta)|}{|\mathbf{m}_j(z)\mathbf{m}_j(\zeta)|}. \tag{5.50}$$

By the lower bound in condition (A), we have $|(S\operatorname{Im}\mathbf{m}(z))_j| \geq c_{inf}|\operatorname{Im} m(z)|$, where $m(z)$ is defined in (3.3) and thus

$$|\mathbf{m}_j(z)| \leq \frac{|\operatorname{Im}\mathbf{m}_j(z)|}{|\mathbf{m}_j(z)|}\cdot\frac{1}{c_{inf}|\operatorname{Im} m(z)| + |\eta|}. \tag{5.51}$$

It follows from the estimate (5.51) and (3.5) that

$$1 - \|F(z,\zeta)\|_{\ell^2\to\ell^2}\operatorname{Re}\langle v, \mathrm{U}v\rangle \geq C\,\|F(z,\zeta)\|_{\ell^2\to\ell^2}\left(c_{inf}|\operatorname{Im} m(z)| + |\eta|\right)\left(c_{inf}|\operatorname{Im} m(\zeta)| + |\eta'|\right). \tag{5.52}$$

Finally, by combining this estimate with (5.47), (5.48) and (3.10) we get

$$\left\|(1 - S\mathbf{m}(z)\mathbf{m}(\zeta))^{-1}\right\|_{\ell^2\to\ell^2} \leq C_\kappa, \tag{5.53}$$

for $z,\zeta \in \mathcal{D}_\kappa$ with $\operatorname{Im} z\operatorname{Im}\zeta > 0$ and some positive constant C_κ dependent on κ.

Proposition 5.7. *Let z,ζ be two spectral parameters in the domain $\mathcal{D}_\kappa$ defined in* (3.9) *with $\operatorname{Re} z$ and $\operatorname{Re}\zeta$ in the same connected component of $\mathcal{I}_\kappa$ as defined in* (3.8). *Then*

$$\left\|\big(1 - \Pi(z)\big)\big(1 - S\mathbf{m}(z)\mathbf{m}(\zeta)\big)^{-1}\right\|_{\ell^2\to\ell^2} \leq c, \tag{5.54}$$

where $c = c_\kappa$ is a positive constant dependent of κ, and $\Pi(z)$ is defined in (4.3).

The proof will requite the following technical lemma.

Lemma 5.8. *Let T be a self-adjoint matrix with positive entries satisfying $\|T\|_{\ell^2\to\ell^2} \leq 1-\varepsilon$. Let v be the ℓ^2-normalized principal eigenvector of T with positive coordinates, and let $\delta := \left\|(1 - vv^*)(1-T)^{-1}\right\|_{\ell^2\to\ell^2}$. Then for any ℓ^2-normalized vector w such that $\|(1-T)w\|_2 \leq \Delta$*

$$\left\|(1 - ww^*)(1-T)^{-1}\right\|_{\ell^2\to\ell^2} \leq \delta\left(1 + \frac{\Delta}{\varepsilon}\right) \tag{5.55}$$

Proof. Denote $A := (1-T)^{-1}$, then $\|A\|_{\ell^2\to\ell^2} \leq 1/\varepsilon$. Since $(1 - vv^*)$ commutes with $(1-T)$ we can write

$$\|(1 - vv^*)w\|_2 = \|A(1 - vv^*)(1-T)w\|_2 \leq \delta\Delta. \tag{5.56}$$

Now by using $I = vv^* + (1 - vv^*)$ we compute

$$\|(1 - ww^*)A\|_{\ell^2\to\ell^2} \leq \|(1 - ww^*)(1 - vv^*)A\|_{\ell^2\to\ell^2} + \|(1 - ww^*)vv^*A\|_{\ell^2\to\ell^2}. \tag{5.57}$$

The first term is bounded by δ, and the second term is bounded by $\|(1 - ww^*)v\|_2/\varepsilon$.
By using $\|v\|_2 = \|w\|_2 = 1$ we get

$$\|(1 - ww^*)v\|_2^2 = \|v\|_2^2 + |\langle v, w\rangle|^2\left(\|w\|_2^2 - 2\right) = 1 - |\langle v, w\rangle|^2. \tag{5.58}$$

It follows that $\|(1 - ww^*)v\|_2 = \|(1 - vv^*)w\|_2 \leq \delta\Delta$, which concludes the proof of the lemma. □

Proof of Proposition 5.7. Fix a parameter z with $\eta := \operatorname{Im} z$ in $\mathcal{D}_\kappa$ and denote the inverse of the stability operator by A to condense the notation.

$$A \equiv A(\zeta) := \big(1 - S\mathbf{m}(z)\mathbf{m}(\zeta)\big)^{-1}. \tag{5.59}$$

By using the identity (5.47) with $\zeta = \bar{z}$ we get

$$A_0 := A(\bar{z}) = |\mathbf{m}(z)|\big(1 - F(z,\bar{z})\big)^{-1}|\mathbf{m}(z)|^{-1}. \tag{5.60}$$

Note that the spectral theorem for the self-adjoint operator $1 - F(z,\bar{z})$ implies that

$$\left\|(1 - vv^*)\big(1 - F(z,\bar{z})\big)^{-1}\right\|_{\ell^2\to\ell^2} \leq \frac{1}{\operatorname{Gap} F(z,\bar{z})}, \tag{5.61}$$

where $v := v(z,\bar{z})$ is the principal eigenvector of $F(z,\bar{z})$.
Furthermore, by taking the imaginary part of the Dyson vector equation (3.2) and multiplying both sides by $|\operatorname{Im}\mathbf{m}_j(z)|$ we get

$$\frac{\operatorname{Im}\mathbf{m}_j(z)}{|\mathbf{m}_j(z)|} = |\mathbf{m}_j(z)|\eta + \sum_{k=1}^{N} |\mathbf{m}_j(z)| S_{jk} \operatorname{Im}\mathbf{m}_k(z), \quad j \in \{1,\dots,N\}. \tag{5.62}$$

This implies, by definition of w in (4.3), that

$$\big(1 - F(z,\bar{z})\big)w = |\eta| \cdot |\mathbf{m}(z)| \left\|\frac{\operatorname{Im}\mathbf{m}(z)}{|\mathbf{m}(z)|}\right\|_2^{-1}. \tag{5.63}$$

Taking the norm of both sides of (5.63) and using (5.51) we get

$$\left\|\big(1 - F(z,\bar{z})\big)w\right\|_2 \leq c|\eta|. \tag{5.64}$$

Applying Lemma 5.8 for $T = F$ and w as in (4.3) and observing that Lemma 5.5 guarantees that $\delta \leq c'$, (3.17) provides $\epsilon \geq c''|\eta|$, and (5.64) gives $\Delta \leq c'''|\eta|$ in the notation of Lemma 5.8, we get

$$\left\|(1 - ww^*)\big(1 - F(z,\bar{z})\big)^{-1}\right\|_{\ell^2\to\ell^2} \leq c. \tag{5.65}$$

We can finally conclude from (5.60) and (3.5) that

$$\left\|\big(1 - \Pi(z)\big)A_0\right\|_{\ell^2\to\ell^2} \leq c. \tag{5.66}$$

To estimate the norm of $(1 - \Pi(z))A$ we write

$$\big(1 - \Pi(z)\big)A = \big(1 - \Pi(z)\big)A_0 + \big(1 - \Pi(z)\big)A_0\big(S\mathbf{m}(z)\left(\mathbf{m}(\bar{z}) - \mathbf{m}(\zeta)\right)\big)A. \tag{5.67}$$

First, we consider the case when ζ and $\bar{z}$ are in the same half-plane, i.e., $\operatorname{Im} z \operatorname{Im}\bar{\zeta} > 0$.
In this regime we can use the upper bound

$$|\mathbf{m}_j(\bar{z}) - \mathbf{m}_j(\zeta)| \leq \int_{\zeta}^{\bar{z}} |\mathbf{m}_j'(\xi)| d\xi. \tag{5.68}$$

Differentiating the vector Dyson equation (3.2) we get

$$\mathbf{m}_j'(\xi) = \sum_k \big(1 - \mathbf{m}(\xi)^2 S\big)^{-1}_{jk} \mathbf{m}_k(\xi)^2 \tag{5.69}$$

Since $\operatorname{Re} z$ and $\operatorname{Re}\zeta$ lie in the same connected component of the support of the self-consistent density of states, so does the projection onto the real axis of the line segment connecting them and by (3.10) for all ξ in $[\zeta,\bar{z}]$ we have

$$\left\|\big(1 - \mathbf{m}(\xi)^2 S\big)^{-1}\right\|_{\ell^\infty\to\ell^\infty} \leq \frac{1}{|\operatorname{Im} m(\xi)|} \leq C. \tag{5.70}$$

Then by using (3.5) we get

$$\|S\mathbf{m}(z)(\mathbf{m}(\bar{z}) - \mathbf{m}(\zeta))\|_{\ell^2\to\ell^2} \leq C|\bar{z} - \zeta|. \tag{5.71}$$

The result then follows by taking the norm of both sides of (5.67) and applying the estimates (5.66), (4.1) and (5.71).
In the other case, when ζ and $\bar{z}$ are in different half-planes, the desired estimate is then a direct consequence of (5.53). □

5.5 Computation of the variance

In turn, this allows us to obtain the following expression

$$
\begin{aligned}
\mathbb{E}\left[\tilde{e}(\lambda)\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\frac{\partial T_{jj}(z,\zeta)}{\partial\zeta}\mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right] &= \\
&= \mathbb{E}\left[\tilde{e}(\lambda)\right]\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\frac{\partial}{\partial\zeta}\left(\sum_{k=1}^{N}\big(1-\Pi(z)\big)_{jk}\left((1-S\mathbf{m}(z)\mathbf{m}(\zeta))^{-1}(S\mathbf{m}(z)\mathbf{m}(\zeta))^2\right)_{kj}\right)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta \\
&+ \mathbb{E}\left[\tilde{e}(\lambda)\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\frac{\partial}{\partial\zeta}\left(\sum_{k=1}^{N}\Pi_{jk}(z)T_{kj}(z,\zeta)\right)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right] + \mathcal{O}_\prec(\Psi(z)^{3/2}\Psi(i\eta_0) + \Psi(z)\Psi(i\eta_0)^{3/2}).
\end{aligned}
$$

Then the assumption (4.2) and Lemmas 5.3, 5.1 yield the following expression.

$$
\begin{aligned}
\mathbb{E}\left[\tilde{e}(\lambda)\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\frac{\partial}{\partial\zeta}\left(\sum_{k=1}^{N}\Pi_{jk}(z)T_{kj}(z,\zeta)\right)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right] &= \mathbb{E}\left[\tilde{e}(\lambda)\right]\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\frac{\partial}{\partial\zeta}\left(\sum_{k=1}^{N}\Pi_{jk}(z)T_{kj}(z,\zeta)\right)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta \\
&+ \mathcal{O}_\prec(N\Theta^2(z) + N\Theta(z)\Theta(i\eta_0)).
\end{aligned} \tag{5.72}
$$

Finally, from Proposition 5.2 we can conclude that

$$
\mathbb{E}\left[\tilde{e}(\lambda)\left\{1-\mathbb{E}\right\}\left[\operatorname{Tr}G(z)\right]\right] = \frac{i\lambda}{\pi}\,\mathbb{E}\left[\tilde{e}(\lambda)\right]\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}}{\partial\bar{\zeta}}\mathcal{K}(z,\zeta)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta + \tilde{\mathcal{E}}(z), \tag{5.73}
$$

where $\mathcal{K}$ is defined in (4.5) and $\tilde{\mathcal{E}}(z)$ is the total error term collected from previous derivations. Hence

$$
\phi'(\lambda) = -\lambda V(f)\,\mathbb{E}\left[\tilde{e}(\lambda)\right] + \tilde{\mathcal{E}}(\lambda),
$$

where

$$
V(f) = \frac{1}{\pi^2}\int\limits_{\Omega_\alpha}\int\limits_{\Omega_\alpha}\frac{\partial\tilde{f}(\zeta)}{\partial\bar{\zeta}}\frac{\partial\tilde{f}(z)}{\partial\bar{z}}\mathcal{K}(z,\zeta)\mathrm{d}\bar{\zeta}\mathrm{d}\zeta. \tag{5.74}
$$

This completes the proof of Theorem 4.1.

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